第五次作业

1. $Map(V\times V, F)$ 是 F 上线性空间，其中加法和数乘定义如下：$f,g\in Map(V\times V,F)$，$\alpha\in F$

$$f+g: V\times V \longrightarrow F \qquad (\vec{x},\vec{y}) \longmapsto f(\vec{x},\vec{y})+g(\vec{x},\vec{y})$$

$$\alpha f: V\times V \longrightarrow F \qquad (\vec{x},\vec{y}) \longmapsto \alpha f(\vec{x},\vec{y})$$

$L_2(V)\subset Map(V\times V, F)$. $L_2^+(V), L_2^-(V)\subset L_2(V)$.

(1) 验证 $L_2(V)$ 是 $Map(V\times V,F)$ 子空间

i.e. $\forall f,g\in L_2(V)$, $\forall \alpha,\beta\in F$. 证明 $\alpha f+\beta g\in L_2(V)$.

$$(\alpha f+\beta g)(\vec{x}+\vec{z},\vec{y}) \underset{\text{加法.数乘定义}}{=} \alpha f(\vec{x}+\vec{z},\vec{y})+\beta g(\vec{x}+\vec{z},\vec{y}) \underset{f.g\in L_2(V)}{=} \alpha\left(f(\vec{x},\vec{y})+f(\vec{z},\vec{y})\right)+\beta\left(g(\vec{x},\vec{y})+g(\vec{z},\vec{y})\right)$$

$$=\left(\alpha f(\vec{x},\vec{y})+\beta g(\vec{x},\vec{y})\right)+\left(\alpha f(\vec{z},\vec{y})+\beta g(\vec{z},\vec{y})\right)$$

$$\underset{\text{加法.数乘定义}}{=}(\alpha f+\beta g)(\vec{x},\vec{y})+(\alpha f+\beta g)(\vec{z},\vec{y})$$

$\forall\lambda\in F$. $(\alpha f+\beta g)(\lambda\vec{x},\vec{y})=\alpha f(\lambda\vec{x},\vec{y})+\beta g(\lambda\vec{x},\vec{y})=\lambda\alpha f(\vec{x},\vec{y})+\lambda\beta g(\vec{x},\vec{y})$

$$=\lambda\left(\alpha f(\vec{x},\vec{y})+\beta g(\vec{x},\vec{y})\right)=\lambda(\alpha f+\beta g)(\vec{x},\vec{y})$$

$\therefore$ $\alpha f+\beta g$ 关于第一个向量是线性的. 同理可证关于第二个向量也是线性的.

$\therefore$ $\alpha f+\beta g\in L_2(V)$

(2). 验证 $L_2^+(V)$, $L_2^-(V)$ 是 $L_2(V)$ 子空间.

$\forall f,g\in L_2^+(V)$. 下证 $\alpha f+\beta g\in L_2^+(V)$.

$\forall\alpha,\beta\in F$.

由 (1) 可知 $\alpha f+\beta g\in L_2(V)$. 只需验证 $\alpha f+\beta g$ 对称即可.

$$(\alpha f+\beta g)(\vec{y},\vec{x}) \underset{\text{加法.数乘定义}}{=} \alpha f(\vec{y},\vec{x})+\beta g(\vec{y},\vec{x}) \underset{f.g\in L_2^+(V)}{=} \alpha f(\vec{x},\vec{y})+\beta g(\vec{x},\vec{y}) \underset{\text{加法.数乘定义}}{=} (\alpha f+\beta g)(\vec{x},\vec{y})$$

$\therefore \alpha f+\beta g \in L_2^+(V)$.

类似可验证 $L_2^-(V)$ 也是 $L_2(V)$ 的子空间.

(3) $char(F)\neq 2$. 证 $L_2(V)=L_2^+(V)\oplus L_2^-(V)$.

设 $f(\vec{x},\vec{y})\in L_2(V)$. 则

$$f^+=\frac{1}{2}(f(\vec{x},\vec{y})+f(\vec{y},\vec{x}))\in L_2^+(V),\quad f^-=\frac{1}{2}(f(\vec{x},\vec{y})-f(\vec{y},\vec{x}))\in L_2^-(V)$$

且 $f=f^++f^-$. $\therefore L_2(V)=L_2^+(V)+L_2^-(V)$.

设 $g\in L_2^+(V)\cap L_2^-(V)$. 则 $g(\vec{x},\vec{y})=g(\vec{y},\vec{x})=-g(\vec{y},\vec{x})$

$$\Rightarrow 2g(\vec{y},\vec{x})=0 \xRightarrow{char(F)\neq 2} g(\vec{y},\vec{x})=0 \Rightarrow g(\vec{x},\vec{y})=0.$$

$\therefore L_2^+(V)\cap L_2^-(V)=\{0\}$. 从而 $L_2(V)=L_2^+(V)\oplus L_2^-(V)$

2. 复习: $A\in M_n(F)$. F 域.

① $\forall \alpha,\beta\in F$, $A,B\in M_n(F)$, $tr(\alpha A+\beta B)=\alpha tr(A)+\beta tr(B)$

② $\forall A,B\in M_n(F)$, $tr(AB)=tr(BA)$.

(1) $f: M_2(F)\times M_2(F)\longrightarrow F$

$$(A,B)\longmapsto tr(AB).$$

f 双线性型: $\forall \alpha,\beta\in F$. $\forall A,B,C\in M_2(F)$.

$$f(\alpha A+\beta C,B)=tr((\alpha A+\beta C)B)=tr(\alpha AB+\beta CB)$$
$$=\alpha tr(AB)+\beta tr(CB)$$
$$=\alpha \underset{f(A,B)}{\cancel{f(AB)}}+\beta \underset{f(C,B)}{\cancel{f(CB)}}.$$

同理可证 $f(A,\alpha B+\beta C)=\alpha f(A,B)+\beta f(A,C)$.

f 对称: $f(A,B)=tr(AB)=tr(BA)=f(B,A)$

(2) 设 $\vec{e}_1 = E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{e}_2 = E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{e}_3 = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\vec{e}_4 = E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

f 在 $\vec{e}_1, \cdots, \vec{e}_4$ 下矩阵为 $A = (f(\vec{e}_i, \vec{e}_j))_{4\times 4}$.

计算可得
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A 可以看成 E_4 交换第2行第3行得到的矩阵. $\therefore \operatorname{rank}(f) = \operatorname{rank}(A) = 4$.

3. (1). $f: \mathbb{R}[x]^{(n)} \times \mathbb{R}[x]^{(n)} \longrightarrow \mathbb{R}$.

$$(p, q) \longmapsto \int_0^1 p(x) q(x)\, dx.$$

f 是双线性型: 由定积分的性质自行验证.

f 对称: $f(p, q) = \int_0^1 p(x) q(x)\, dx = \int_0^1 q(x) p(x)\, dx = f(q, p)$.

(2). 设 $\vec{e}_1 = 1, \vec{e}_2 = x, \cdots, \vec{e}_n = x^{n-1}$.

f 在 $\vec{e}_1, \cdots, \vec{e}_n$ 下矩阵为 $A = (f(\vec{e}_i, \vec{e}_j))_{n\times n}$

计算可得.
$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}_{n\times n}$$

下证 A 是满秩的. 想法: 利用以 A 为系数矩阵的齐次线性方程组只有零解说明 A 满秩.

设 $p = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1} = (1, x, \cdots, x^{n-1}) \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{pmatrix}$

$q = q_0 + q_1 x + \cdots + q_{n-1} x^{n-1} = (1, x, \cdots, x^{n-1}) \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{pmatrix}$

$$f: \mathbb{R}[x]^{(n)} \times \mathbb{R}[x]^{(n)} \longrightarrow \mathbb{R}.$$

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{pmatrix}, \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{pmatrix} \longmapsto (p_0, p_1, \cdots, p_{n-1}) A \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{pmatrix}$$

考虑齐次线性方程组 $A\vec{y}=\vec{0}$.

设 $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})^t$ 是 $A\vec{y}=\vec{0}$ 的解. 则 $A\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

令 $\alpha(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$.

则 $f(\alpha, \alpha) = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = 0$. i.e. $\int_0^1 \alpha^2(x)\,dx = 0$.

由定积分的性质可知 $\alpha(x)=0$. $\therefore \alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$.

这意味着 $A\vec{y}=\vec{0}$ 只有零解. $\therefore \mathrm{rank}(A)=n$. $\Rightarrow \mathrm{rank}(f)=n$.

4. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. 计算 $P \in GL_n(\mathbb{Q})$ s.t. $P^tAP=B$ 是对角阵.

法一: 行列相伴变换方法.

$$(A|E_3) = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[c_1 \leftrightarrow c_2]{r_1 \overset{换}{\leftrightarrow} r_2} \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[c_1 加到 c_2]{r_1 加到 r_2} \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[c_1 加到 c_3]{r_1 加到 r_3} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[c_2 \times (-1) 加到 c_3]{r_2 \times (-1) 加到 r_3} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

左边 3×3 块为 B，右边 3×3 块为 P^t.

$\therefore P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P^tAP = B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

法二：降维法.

取$\mathbb{R}^3$标准基$\vec{e}_1,\vec{e}_2,\vec{e}_3$. $\vec{x},\vec{y}\in\mathbb{R}^3$. $\vec{x}=(\vec{e}_1,\vec{e}_2,\vec{e}_3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $\vec{y}=(\vec{e}_1,\vec{e}_2,\vec{e}_3)\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

设 $f:\mathbb{R}^3\times\mathbb{R}^3\longrightarrow\mathbb{R}$. 则 $f\in L_2^+(\mathbb{R}^3)$. 且 A 是 f 在 $\vec{e}_1,\vec{e}_2,\vec{e}_3$ 下的矩阵.

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\longmapsto(x_1,x_2,x_3)A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}.$$

i.e. $A=\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}=(f(\vec{e}_i,\vec{e}_j))_{3\times3}$.

①选取 $\vec{\varepsilon}_1$ s.t. $f(\vec{\varepsilon}_1,\vec{\varepsilon}_1)\neq0$. 令 $\vec{\varepsilon}_1=\vec{e}_2$. $f(\vec{\varepsilon}_1,\vec{\varepsilon}_1)=-1\neq0$.

②确定 $W=\ker(f(\vec{x},\vec{\varepsilon}_1))$ 的一组基.

$$f(\vec{x},\vec{\varepsilon}_1)=(x_1,x_2,x_3)A\begin{pmatrix}0\\1\\0\end{pmatrix}=x_1-x_2+x_3.$$

解方程 $x_1-x_2+x_3=0$ 得 W 一组基. $\vec{w}_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$, $\vec{w}_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$

③求 $g:=f|_{W\times W}$ 在 $\vec{w}_1,\vec{w}_2$ 下的矩阵.

$$B=\begin{pmatrix}f(\vec{w}_1,\vec{w}_1) & f(\vec{w}_1,\vec{w}_2)\\ f(\vec{w}_2,\vec{w}_1) & f(\vec{w}_2,\vec{w}_2)\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

到此降维到 W 上的对称双线性型 g.

①选取 $\vec{\varepsilon}_2$ s.t. $g(\vec{\varepsilon}_2,\vec{\varepsilon}_2)\neq0$. 令 $\vec{\varepsilon}_2=\vec{w}_2$. $g(\vec{w}_2,\vec{w}_2)=f(\vec{w}_2,\vec{w}_2)=1\neq0$.

②确定 $Z=\ker(g(\vec{y},\vec{\varepsilon}_2))$ 一组基.

$$g(\vec{y},\vec{\varepsilon}_2)=(y_1,y_2)B\begin{pmatrix}0\\1\end{pmatrix}=y_2.$$

解方程 $y_2=0$ 得解空间一组基: $\begin{pmatrix}1\\0\end{pmatrix}$ $\Rightarrow$ Z 基是 $(\vec{w}_1,\vec{w}_2)\begin{pmatrix}1\\0\end{pmatrix}$

于是. f 在 $\mathbb{R}^3$ 的一组规范基是

$$\vec{\varepsilon}_1=\vec{e}_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \vec{\varepsilon}_2=\vec{w}_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\quad \vec{\varepsilon}_3=\vec{w}_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

从 $\vec{e}_1,\vec{e}_2,\vec{e}_3$ 到 $\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{\varepsilon}_3$ 的矩阵为 $P=\begin{pmatrix}0&1&-1\\1&1&0\\0&0&1\end{pmatrix}$

计算得. $P^tAP=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

法三：配方法.

取$\mathbb{R}^3$的标准基e_1, e_2, e_3

$$A=\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$ 对应的$\mathbb{R}^3$上二次型为 $q=-x_2^2+2x_1x_2+2x_2x_3$

$$=(x_1,x_2,x_3)A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

$$q=-x_2^2+2x_1x_2+2x_2x_3$$

$$=-(x_1-x_2+x_3)^2+x_1^2+x_3^2+2x_1x_3$$

$$=-(x_1-x_2+x_3)^2+(x_1+x_3)^2.$$

令 $\begin{cases}y_1=x_1-x_2+x_3\\y_2=x_1+x_3\\y_3=x_3.\end{cases}$ i.e. $\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\underbrace{\begin{pmatrix}1&-1&1\\1&0&1\\0&0&1\end{pmatrix}}_{Q}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

计算 $Q^{-1}=\begin{pmatrix}0&1&-1\\-1&1&0\\0&0&1\end{pmatrix}\triangleq P$. 则 $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=P\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

则 $q=-y_1^2+y_2^2$，且.

$$q=(x_1,x_2,x_3)A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=(y_1,y_2,y_3)P^tAP\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=(y_1,y_2,y_3)\begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

于是 $A\sim_c\underbrace{\begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}}_{B}$ 且规范基是 $P=\begin{pmatrix}0&1&-1\\-1&1&0\\0&0&1\end{pmatrix}$ 的三个列向量.

i.e. $P^tAP=B$