第七次作业.

1. (1) 令 $A=\begin{pmatrix}1 & -1\\ -1 & 3\end{pmatrix}$ $\Delta_1=1>0$. $\Delta_2=|A|=3-1=2>0$ $\Rightarrow$ A 正定.

(2) 令 $A=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$ $|A|=3-4=-1<0$. $\Rightarrow$ A 非正定.

2. 实二次型 $\lambda x_1^2-2x_2^2-3x_3^2+2x_1x_2-2x_1x_3+2x_2x_3$ 在 λ 取什么值时呈负定的.

解: q 在标准基下矩阵为 $A=\begin{pmatrix}\lambda & 1 & -1\\ 1 & -2 & 1\\ -1 & 1 & -3\end{pmatrix}$.

A 负定 $\Leftrightarrow B=-A=\begin{pmatrix}-\lambda & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 3\end{pmatrix}$ 正定

设 B 的 i 阶顺序主子式为 Δ_i, $i=1,2,3$.

$$\begin{cases}\Delta_1=-\lambda>0\\ \Delta_2=\begin{vmatrix}-\lambda & -1\\ -1 & 2\end{vmatrix}=-2\lambda-1>0\\ \Delta_3=|B|=-5\lambda-3>0\end{cases}\Rightarrow \lambda<-\frac{3}{5}$$

3. $A\in M_n(\mathbb{R})$. 证:

(1) $A\in SSM_n(\mathbb{R}) \Leftrightarrow \forall \vec{x}\in\mathbb{R}^n$. $\vec{x}^tA\vec{x}=0$.

(2) 若 $A\in SM_n(\mathbb{R})$. 且 $\forall\vec{x}\in\mathbb{R}^n$, $\vec{x}^tA\vec{x}=0$. 则 $A=O$.

Pf: (1) "$\Longrightarrow$" $A^t=-A$. 则 $\forall\vec{x}\in\mathbb{R}^n$.

$$\vec{x}^tA\vec{x}=(\vec{x}^tA\vec{x})^t=\vec{x}^tA^t\vec{x}=\vec{x}^t(-A)\vec{x}=-\vec{x}^tA\vec{x}.$$

$\therefore 2\vec{x}^tA\vec{x}=0$. $\because char(\mathbb{R})\neq 2$ $\therefore \vec{x}^tA\vec{x}=0$.

"$\Longleftarrow$" 设 $A=(a_{ij})_{n\times n}$. 则对 $\forall\vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in\mathbb{R}^n$.

$$0=\vec{x}^tA\vec{x}=\sum_{i=1}^n a_{ii}x_i^2+\sum_{1\le i<j\le n}(a_{ij}+a_{ji})x_ix_j \qquad (*)$$

分别令 $\vec{x}=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\leftarrow$ 第 i 行为 1, 其余为 0. $=\underset{\substack{\uparrow\\ \text{标准基第 i 个}}}{\vec{e_i}}$, $i=1,\dots,n$. 代入 (*) 可得.

$a_{ii}=0$. $i=1,\cdots,n$. $\Rightarrow \sum\limits_{1\le i<j\le n}(a_{ij}+a_{ji})x_ix_j=0$. (**)

再分别令 $\vec{x}=\begin{array}{c}\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\end{array}\begin{array}{l}\\ \\ \leftarrow i\\ \\ \leftarrow j\\ \\ \\ \end{array}=\vec{e}_i+\vec{e}_j$. , $1\le i<j\le n$. 代入(**)可得.

$a_{ij}+a_{ji}=0$. i.e. $a_{ij}=-a_{ji}$. $\forall\, 1\le i<j\le n$.

$\Rightarrow A=-A^t$ $\therefore A$为斜对称矩阵.

(2) 法一: 由(1)可知. $A\in SSM_n(\mathbb{R})$. $\therefore A^t=-A$.

又$\because A\in SM_n(\mathbb{R})$ $\therefore A=A^t=-A$. $\xRightarrow{char(\mathbb{R})\neq 2}$ $A=0$.

法二: 设 $q:\mathbb{R}^n\longrightarrow\mathbb{R}$ 是 A对应的二次型, A是q在标准基下的矩阵.

$\vec{x}\longmapsto\vec{x}^tA\vec{x}$

$\because \forall\vec{x}\in\mathbb{R}^n$, $q(\vec{x})=0$ 且 q在标准基下的矩阵唯一. $\therefore A=0$.

4. $A\in M_n(\mathbb{R})$. 证:

(1) $q(A)=tr(A^tA)$为$M_n(\mathbb{R})$上的正定二次型;

(2) 若$A^tA=A^2$, 则$A\in SM_n(\mathbb{R})$.

Pf: (1) 先证q是$M_n(\mathbb{R})$上二次型. 设$B\in M_n(\mathbb{R})$.

① $q(-A)=tr((-A)^t(-A))=tr((-A^t)(-A))=tr(A^tA)=q(A)$.

② 令 $f(A,B)=\frac{1}{2}(q(A+B)-q(A)-q(B))$

$=\frac{1}{2}[tr((A+B)^t(A+B))-tr(A^tA)-tr(B^tB)]$ (注意 $(A+B)^t=A^t+B^t$)

$=\frac{1}{2}[tr(A^tA+A^tB+B^tA+B^tB)-tr(A^tA)-tr(B^tB)]$

$=\frac{1}{2}[tr(A^tB+B^tA)]$

$=\frac{1}{2}[tr(A^tB)+tr(B^tA)]$ $(=f(B,A))$

由tr是$M_n(\mathbb{R})$上的线性函数可知f是$M_n(\mathbb{R})$上的双线性型.

对称: $f(A,B)=\frac{1}{2}[tr(A^tB)+tr(B^tA)]=f(B,A)$. $\therefore f$是对称双线性型.

下证 q 是正定的.

$\forall A\in M_n(\mathbb{R})$. $A\neq 0$. $\overset{\text{设}A=(a_{ij})}{\frown}$ $\exists\, 1\le i_0,j_0\le n$ s.t. $a_{i_0j_0}\neq 0$. 则

$$tr(A^tA)=tr\begin{pmatrix}\sum_{i=1}^n a_{i1}^2 & & * \\ & \sum_{i=1}^n a_{i2}^2 & \\ & & \ddots \\ * & & \sum_{i=1}^n a_{in}^2\end{pmatrix}=\sum_{j=1}^n\sum_{i=1}^n a_{ij}^2>0.$$

$\therefore$ q 是 $M_n(\mathbb{R})$ 上的正定二次型.

(2). $q(A-A^t)=tr\left((A-A^t)^t(A-A^t)\right)=tr\left((A^t-A)(A-A^t)\right)$

$$=tr(A^tA-A^2-A^tA^t+AA^t)\overset{A^tA=A^2}{=\!=\!=}tr(AA^t-A^tA^t)$$

$$=tr(AA^t)-tr(A^tA^t)$$

$\because$ $A^tA^t=(A\cdot A)^t=(A^2)^t=(A^tA)^t=A^tA$ 且 $tr(A^tA)=tr(AA^t)=\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2$

(设 $A=(a_{ij})$)

$\therefore$ $q(A-A^t)=tr(AA^t)-tr(A^tA)=0$.

由于 q 是正定二次型 $\therefore$ $A-A^t=0$. i.e. $A^t=A$.

5. 设 $A\in SM_n(\mathbb{R})$. 证: $\exists\,\delta>0$ s.t. $\forall\,\varepsilon\in(-\delta,\delta)$, $E+\varepsilon A$ 正定.

Pf: 设 $A=(a_{ij})_{n\times n}$, A_k 是 A 的前 k 行和前 k 列组成的矩阵, $k=1,2,\cdots,n$.

则 $E+\varepsilon A$ 的 k 阶顺序主子式为

$$\Delta_k(\varepsilon)=|E_k+\varepsilon A_k|=\begin{vmatrix}1+\varepsilon a_{11} & \varepsilon a_{12} & \cdots & \varepsilon a_{1k}\\ \varepsilon a_{21} & 1+\varepsilon a_{22} & \cdots & \varepsilon a_{2k}\\ \vdots & \vdots & & \vdots\\ \varepsilon a_{k1} & \varepsilon a_{k2} & \cdots & 1+\varepsilon a_{kk}\end{vmatrix}\overset{\text{记为}}{=\!=}C_{kk}\varepsilon^k+C_{k,k-1}\varepsilon^{k-1}+\cdots+C_{k1}\varepsilon+1$$

$(C_{ij}\in\mathbb{R})$

$\Delta_k(\varepsilon)$ 是关于 ε 的多项式. $\therefore$ 它是关于 ε 的连续函数.

$\because$ $\Delta_k(0)=1$. $\therefore$ $\exists\,\delta_k>0$ s.t. $\forall\,\varepsilon\in(-\delta_k,\delta_k)$, $\Delta_k(\varepsilon)>0$.

令 $\delta=\min\{\delta_1,\cdots,\delta_n\}$. 则当 $\varepsilon\in(-\delta,\delta)$ 时, $\Delta_k(\varepsilon)>0$, $k=1,2,\cdots,n$.

$\Rightarrow$ 当 $\varepsilon\in(-\delta,\delta)$ 时, $E+\varepsilon A$ 正定.

例1.6. 设 $A\in F^{m\times n}$. $\varphi: F^{n\times k}\to F^{m\times k}$, $X\mapsto AX$. 求 φ 在标准基下的矩阵.

解：由矩阵乘法性质知 $\varphi\in Hom(F^{n\times k}, F^{m\times k})$.

设 $X=(\vec{x}^{(1)},\vec{x}^{(2)},\cdots,\vec{x}^{(k)})$，则 $AX=(A\vec{x}^{(1)},A\vec{x}^{(2)},\cdots A\vec{x}^{(k)})$.

注意到：$F^{n\times k}$（矩阵空间）与 F^{nk}（向量空间）是线性同构.

$$\varphi: F^{n\times k}\longrightarrow F^{m\times k}$$

$$(\vec{x}^{(1)},\vec{x}^{(2)},\cdots,\vec{x}^{(k)})\longmapsto (A\vec{x}^{(1)},A\vec{x}^{(2)},\cdots,A\vec{x}^{(k)}) = (\overrightarrow{\varphi(X)}^{(1)},\overrightarrow{\varphi(X)}^{(2)},\cdots,\overrightarrow{\varphi(X)}^{(k)})$$

$$\psi: F^{nk}\longrightarrow F^{mk}$$

$$\begin{pmatrix}\vec{x}^{(1)}\\ \vec{x}^{(2)}\\ \vdots\\ \vec{x}^{(k)}\end{pmatrix}\longmapsto\begin{pmatrix}A\vec{x}^{(1)}\\ A\vec{x}^{(2)}\\ \vdots\\ A\vec{x}^{(k)}\end{pmatrix}$$

$\because$ $\overrightarrow{\varphi(X)}^{(j)}=A\vec{x}^{(j)}$, $j=1,2,\cdots,k$.

$\therefore$

$$\begin{pmatrix}\overrightarrow{\varphi(X)}^{(1)}\\ \overrightarrow{\varphi(X)}^{(2)}\\ \vdots\\ \overrightarrow{\varphi(X)}^{(k)}\end{pmatrix}=\underbrace{\begin{pmatrix}A_{m\times n} & O & \cdots & O\\ O & A_{m\times n} & \cdots & O\\ \vdots & \vdots & & \vdots\\ O & O & \cdots & A_{m\times n}\end{pmatrix}_{km\times kn}}_{=B}\begin{pmatrix}\vec{x}^{(1)}\\ \vec{x}^{(2)}\\ \vdots\\ \vec{x}^{(k)}\end{pmatrix}$$

设 F^{nk} 标准基：$\vec{e}_1,\vec{e}_2,\vec{e}_3,\cdots,\vec{e}_n,\cdots\cdots\vec{e}_{nk}$

$F^{n\times k}$ 标准基：$E_{11},E_{21},E_{31},\cdots,E_{n1},\cdots,E_{1,k},\cdots,E_{n,k}$

F^{mk} 标准基：$\vec{\varepsilon}_1,\vec{\varepsilon}_2,\cdots\cdots\vec{\varepsilon}_m,\cdots\cdots\vec{\varepsilon}_{mk}$

$F^{m\times k}$ 标准基：$L_{11},L_{21},\cdots\cdots L_{m1},\cdots L_{1k},\cdots L_{m,k}$.

（$F^{n\times k}$, $F^{m\times k}$ 标准基排列方式：列优先，列小的排前面，列一样时行小的排前面）

B 是 ψ 在 $\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_{nk}$；$\vec{\varepsilon}_1,\vec{\varepsilon}_2,\cdots,\vec{\varepsilon}_{mk}$ 下的矩阵.

B 也是 φ 在 $E_{11},E_{21},\cdots,E_{n1},E_{1,k},\cdots,E_{n,k}$；$L_{11},L_{21},\cdots L_{m1},\cdots,L_{1k},\cdots,L_{m,k}$ 下的矩阵.

Note:

设V, W是线性空间，基底分别为 $\vec{e}_1, \cdots, \vec{e}_n$; $\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m$

设 $\alpha: V \to V'$, $\beta: W \to W'$ 是线性同构. 设 $\varphi \in Hom(V, W)$. 则

(1) $\psi = \beta \circ \varphi \circ \alpha^{-1} \in Hom(V', W')$

(2) ψ在 $\alpha(\vec{e}_1), \cdots, \alpha(\vec{e}_n)$; $\beta(\vec{\varepsilon}_1), \cdots, \beta(\vec{\varepsilon}_m)$ 下的矩阵与

φ在 $\vec{e}_1, \cdots, \vec{e}_n$; $\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m$ 下的矩阵相同.

证: (1)

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ V' \arrow[r, "\overbrace{\beta\circ\varphi\circ\alpha^{-1}}^{\psi}"] & W' \end{tikzcd}$$

可验证 $\psi \in Hom(V', W')$

(2) 设 $(\varphi(\vec{e}_1), \cdots, \varphi(\vec{e}_n)) = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) A$

则 $\psi(\alpha(\vec{e}_1), \cdots, \alpha(\vec{e}_n))$

$= (\beta\circ\varphi\circ\alpha^{-1}\circ\alpha(\vec{e}_1), \cdots, \beta\circ\varphi\circ\alpha^{-1}\circ\alpha(\vec{e}_n))$

$= (\beta(\varphi(\vec{e}_1)), \cdots, \beta(\varphi(\vec{e}_n)))$

$= \beta(\varphi(\vec{e}_1), \cdots, \varphi(\vec{e}_n))$

$= \beta((\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) A)$

$= (\beta(\vec{\varepsilon}_1), \cdots, \beta(\vec{\varepsilon}_m)) A$. [见李老师讲义引理1.7]

现在考虑矩阵空间

$$vec: F^{m\times n} \longrightarrow F^{mn} \qquad \text{是线性同构}$$
$$A \longmapsto \begin{pmatrix} \vec{A}^{(1)} \\ \vdots \\ \vec{A}^{(n)} \end{pmatrix}$$

vec 称为"矩阵的向量化".

例：设 $m=n=k=2$.

$$\varphi: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$X \longmapsto AX$$

$$\psi = vec \circ \varphi \circ vec^{-1}: \mathbb{R}^4 \longrightarrow \mathbb{R}^4.$$

$\psi(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)M$. 其中 $M = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in M_4(\mathbb{R})$.

注意到：$vec^{-1}(\vec{e}_1) = E_{1,1}$, $vec^{-1}(\vec{e}_2) = E_{2,1}$,
$vec^{-1}(\vec{e}_3) = E_{1,2}$, $vec^{-1}(\vec{e}_4) = E_{2,2}$.

于是. $\varphi(E_{1,1}, E_{2,1}, E_{1,2}, E_{2,2}) = (E_{1,1}, E_{2,1}, E_{1,2}, E_{2,2})M$.

例如：$\varphi(E_{1,1}) = a_{1,1}E_{1,1} + a_{2,1}E_{2,1} = \begin{pmatrix} a_{1,1} & 0 \\ a_{2,1} & 0 \end{pmatrix}$. $\quad A = (a_{ij})_{2\times 2}$