第十八次习题课

一、作业中的问题

1\. (i) $\mathscr{A}(\alpha\vec{x}+\beta\vec{y})=\alpha\mathscr{A}(\vec{x})+\beta\mathscr{A}(\vec{y})\quad \forall\alpha,\beta\in\mathbb{R},\ \vec{x},\vec{y}\in V.$

(ii) 保内积: $(\vec{x}|\vec{y})=(\mathscr{A}(\vec{x})|\mathscr{A}(\vec{y}))$

保距: $\|\vec{x}-\vec{y}\|=\|\mathscr{A}(\vec{x})-\mathscr{A}(\vec{y})\|$

保长: $(\mathscr{A}(\vec{x})|\mathscr{A}(\vec{x}))=(\vec{x}|\vec{x})$.

由课上讲义 6.9 可知只需证一个即可.

(iii) $n=1$ 时, $\mathscr{A}$ 的特征根只有 -1.

$n>1$ 时. $\vec{v}$ 扩充为 V 的一组单位正交基 $\vec{v},\vec{\varepsilon}_2,\cdots,\vec{\varepsilon}_n$.

$$\mathscr{A}(\vec{v})=\vec{v}-2(\vec{v}|\vec{v})\vec{v}=\vec{v}-2\vec{v}=-\vec{v}.$$

$\lambda_1=-1$ $\quad\vec{v}$ 是其对应特征向量.

$$\mathscr{A}(\vec{\varepsilon}_i)=\vec{\varepsilon}_i-2\underbrace{(\vec{\varepsilon}_i|\vec{v})}_{=0\ \text{(单位正交)}}\vec{v}=\vec{\varepsilon}_i\quad=\vec{\varepsilon}_i.$$

$\lambda_2=1$ $\quad\vec{\varepsilon}_2,\cdots,\vec{\varepsilon}_n$ 是特征向量.

$\mathscr{A}$ 的特征根 ±1.

$n>1$ 时 也可以 $\mathscr{A}^2=\mathscr{E}$ 且 $\mathscr{A}$ 非数乘算子

得到 $\mu_{\mathscr{A}}(t)=(t-1)(t+1)$.

H-C 加强版 $\quad\operatorname{spec}_{\mathscr{A}}=\{-1,1\}$

2\. 解:

$$E-A=\begin{bmatrix}-1&-2&2\\-2&-4&4\\2&4&-4\end{bmatrix}\qquad \operatorname{rank}(E-A)=1.$$

V^1 是 $x_1+2x_2-2x_3=0$ 的解空间.

$$V^1=\left\langle\underbrace{\begin{pmatrix}2\\0\\1\end{pmatrix}}_{\vec{u}_1},\underbrace{\begin{pmatrix}2\\-1\\0\end{pmatrix}}_{\vec{u}_2}\right\rangle\xrightarrow{\text{Gram-Schmidt}}=\left\langle\underbrace{\begin{pmatrix}\frac{2}{5}\sqrt{5}\\0\\\frac{1}{5}\sqrt{5}\end{pmatrix}}_{\vec{\varepsilon}_1},\underbrace{\begin{pmatrix}\frac{2}{15}\sqrt{5}\\-\frac{1}{3}\sqrt{5}\\\frac{-4\sqrt{5}}{15}\end{pmatrix}}_{\vec{\varepsilon}_2}\right\rangle.$$

$\mathbb{R}^3 = V^1 \oplus V^{10}$ $\Rightarrow V^{10} = (V^1)^\perp$

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$$V^{10} = \left\langle \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\rangle = \langle \underbrace{\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}}_{\vec{\varepsilon_3}} \rangle.$$

$P = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3})$ $D = P^t A P = \mathrm{diag}(1, 1, 10)$.

若 $\chi_A = (t-\lambda_1)(t-\lambda_2)(t-\lambda_3)$ $\lambda_1, \lambda_2, \lambda_3$ 彼此不相同.

$\mathrm{rank}(\lambda_1 E - A) \Rightarrow V^{\lambda_1} = \langle \vec{u_1} \rangle$.

$\mathrm{rank}(\lambda_2 E - A) \Rightarrow V^{\lambda_2} = \langle \vec{u_2} \rangle$

$\mathrm{rank}(\lambda_3 E - A) \Rightarrow V^{\lambda_3} = \langle \vec{u_3} \rangle$

$$\vec{u_1}, \vec{u_2}, \vec{u_3} \xrightarrow[\text{Gram-Schmidt}]{} \vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}.$$

4. 证：对n归纳

$n=1$时，$A = (\pm 1)$ ✓

$n>1$时且$n-1$时结论成立

$\because$ A是上三角矩阵

$$\therefore \quad A = \begin{pmatrix} a & C \\ O_{(n-1)\times 1} & B \end{pmatrix}$$ $a \in \mathbb{R}$ $B \in M_{n-1}(\mathbb{R})$上三角 $C \in \mathbb{R}^{1\times(n-1)}$.

$\because$ A正交

$\therefore$ A正规.

引理5.11 得 $A = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}$.

$$E = A^t A = \begin{pmatrix} a^2 & 0 \\ 0 & B^t B \end{pmatrix}$$

$a^2 = 1$且B正交．再利用归纳假设可得。 $\square$

5. 证：设 $\mathscr{A}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x \longmapsto A\vec{x}$.

$\because$ A对称

$\therefore$ $\mathscr{A}$ 为对称算子

$\mathbb{R}^n$中有一组单位正交基 $\vec{\varepsilon_1}, \cdots, \vec{\varepsilon_n}$ s.t. $\mathscr{A}$在该基下的矩阵 $B=diag(\lambda_1, \cdots \lambda_n)$

其中$\lambda_1, \cdots, \lambda_n$是A的所有特征根，不必两两不同。

设 $\vec{x} = x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$ 则

$$(A\vec{x} | \vec{x}) = (\mathscr{A}(\vec{x}) | \vec{x}) = \left(\sum_{i=1}^n (\lambda_i x_i)\vec{\varepsilon_i} \,\middle|\, \sum_{i=1}^n x_i\vec{\varepsilon_i}\right) = \sum_{i=1}^n \lambda_i x_i^2.$$

利用$\lambda = \max\{\lambda_1, \cdots, \lambda_n\}$和上式

$$(A\vec{x} | \vec{x}) \leq \lambda \sum_{i=1}^n x_i^2 = \lambda(\vec{x} | \vec{x}).$$

等号成立. $(\mathscr{A}(\vec{x}) | \vec{x}) = \lambda(\vec{x} | \vec{x}) \Longleftrightarrow x \in V^\lambda$.

"$\Longleftarrow$" $\vec{x} \in V^\lambda$ 则 $\mathscr{A}(\vec{x}) = \lambda\vec{x} \Longrightarrow (\mathscr{A}(\vec{x}) | \vec{x}) = (A\vec{x} | \vec{x})$

$$= (\lambda\vec{x} | \vec{x})$$
$$= \lambda(\vec{x} | \vec{x}).$$

"$\Longrightarrow$" $(\mathscr{A}(\vec{x}) | \vec{x}) = \lambda(\vec{x} | \vec{x}) \Longleftrightarrow \underbrace{(\lambda - \lambda_1)}_{\geq 0} x_1^2 + \cdots + \underbrace{(\lambda - \lambda_n)}_{\geq 0} x_n^2 = 0$

$$\Longleftrightarrow (\lambda - \lambda_i) x_i^2 = 0 \quad i = 1, 2, \cdots, n$$

$$\vec{x} = x_1\vec{\varepsilon_1} + \cdots + x_n\vec{\varepsilon_n}$$

若 $x_i \neq 0$ 则 $\lambda = \lambda_i$

即 $\vec{x} = x_{i_1}\vec{\varepsilon_{i_1}} + \cdots + x_{i_k}\vec{\varepsilon_{i_k}}$ $\quad \{i_1, \cdots i_k\} \in \{1, 2, \cdots, n\}$

$x_{i_1} \neq 0, \cdots, x_{i_k} \neq 0$

$\lambda = \lambda_{i_1} = \cdots = \lambda_{i_k}$

i.e. $\mathscr{A}(\vec{\varepsilon_{i_j}}) = \lambda\vec{\varepsilon_{i_j}}$

$$\mathscr{A}(\vec{x}) = x_{i_1}\mathscr{A}(\vec{\varepsilon_{i_1}}) + \cdots + x_{i_k}\mathscr{A}(\vec{\varepsilon_{i_k}})$$
$$= \lambda(x_{i_1}\vec{\varepsilon_{i_1}} + \cdots + x_{i_k}\vec{\varepsilon_{i_k}})$$
$$= \lambda\vec{x}$$

$\Longrightarrow x \in V^\lambda$.

关于正规算子的补充

引理1：设 $\mathscr{A} \in \mathcal{L}(V)$. 则 $(\mathscr{A}^*)^* = \mathscr{A}$.

证：设 $\vec{e}_1, \cdots \vec{e}_n$ 是V的一组单位正交基，A是 $\mathscr{A}$ 在该基下矩阵.

则 A^t 是 $\mathscr{A}^*$ ……

同理 $(A^t)^t$ $(\mathscr{A}^*)^*$ ……

于是 $(\mathscr{A}^*)^* = \mathscr{A}$.

引理2. 设 $\mathscr{A} \in \mathcal{L}(V)$ 正规. 如果 $U \subset V$ 是 $\mathscr{A}$-子空间，则U也是 $\mathscr{A}^*$-子空间.

证：由课上引理5.12. $U^\perp$ 是 $\mathscr{A}$-子空间.

设 $\vec{e}_1, \cdots, \vec{e}_k$ 是V的一组单位正交基。则 $U^\perp$ 有一组单位正交基 $\vec{e}_{k+1}, \cdots \vec{e}_n$

且 $\vec{e}_1, \cdots \vec{e}_k, \vec{e}_{k+1}, \cdots, \vec{e}_n$ 是V的一组单位正交基.

$\mathscr{A}$ 在该基下的矩阵是

$$A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$$

其中 $A_1 \in M_k(\mathbb{R})$, $A_2 \in M_{n-k}(\mathbb{R})$.

于是 $\mathscr{A}^*$ 在该基下的矩阵是

$$A^t = \begin{pmatrix} A_1^t & \\ & A_2^t \end{pmatrix}.$$

由此得出 $(\mathscr{A}^*(\vec{e}_1), \cdots, \mathscr{A}^*(\vec{e}_k)) = (\vec{e}_1, \cdots, \vec{e}_k) A_1^t$

故U是 $\mathscr{A}^*$-子空间.

引理3. 设 $\mathscr{A} \in \mathcal{L}(V)$ 正规，$\lambda \in \operatorname{spec}_{\mathbb{R}}(\mathscr{A})$. 则 $\lambda \in \operatorname{spec}_{\mathbb{R}}(\mathscr{A}^*)$ 且关于 $\mathscr{A}$ 和 λ 的特征子空间 $V_{\mathscr{A}}^{\lambda}$ 等于关于 $\mathscr{A}^*$ 和 λ 的特征子空间 $V_{\mathscr{A}^*}^{\lambda}$.

证：设 $\vec{v} \in V_{\mathscr{A}}^{\lambda}$. 则 $\langle \vec{v} \rangle$ $\mathscr{A}$-不变 $\Rightarrow \langle \vec{v} \rangle$ $\mathscr{A}^*$ 不变. 故 $\exists \alpha$. $\mathscr{A}^*(\vec{v}) = \alpha \vec{v}$.

$\because \mathscr{A}(\vec{v}) = \lambda \vec{v}$ $\therefore (\vec{v} \mid \mathscr{A}(\vec{v})) = (\vec{v} \mid \lambda \vec{v}) = \lambda (\vec{v} \mid \vec{v})$. 另一方面

$(\vec{v} \mid \mathscr{A}(\vec{v})) = (\mathscr{A}^*(\vec{v}) \mid \vec{v}) = \alpha (\vec{v} \mid \vec{v})$. $\Rightarrow \lambda = \alpha$.

我们得到 $\lambda \in \operatorname{spec}_{\mathbb{R}}(\mathscr{A}^*)$ 且 $\vec{v} \in V_{\mathscr{A}^*}^{\lambda}$ $\Rightarrow$ $V_{\mathscr{A}}^{\lambda} \subset V_{\mathscr{A}^*}^{\lambda}$, $V_{\mathscr{A}^*}^{\lambda} \subset V_{\mathscr{A}^{**}}^{\lambda}$ $\Rightarrow V_{\mathscr{A}}^{\lambda} = V_{\mathscr{A}^*}^{\lambda}$.

□.

设$\mathscr{A}\in\mathscr{L}(V)$是正规算子。证明:

(i) $\ker(\mathscr{A})=\ker(\mathscr{A}^*)$

(ii) $\ker(\mathscr{A})^\perp=\mathrm{im}(\mathscr{A})$

(iii) $\mathrm{im}(\mathscr{A})=\mathrm{im}(\mathscr{A}^*)$

证:(i) 法一:设$\mathscr{A}$在V的某组单位正交基$\vec{e_1},\cdots,\vec{e_n}$下的矩阵是$A$.

则$\mathscr{A}^*$在同样基底下的矩阵是A^t.

$\because \mathrm{rank}(A)=\mathrm{rank}(A^t)$

$\therefore \mathrm{rank}(\mathscr{A})=\mathrm{rank}(\mathscr{A}^*)$.

于是$\mathscr{A}$可逆时,$\mathscr{A}^*$可逆.

a. $\ker(\mathscr{A})=\{0\}\Longrightarrow\ker(\mathscr{A}^*)=\ker(\mathscr{A})$.

b. $\ker(\mathscr{A})\neq\{0\}$ [设$\mathscr{A}\in\mathscr{L}(V)$正规,$\lambda\in\mathrm{spec}_{\mathbb{R}}(\mathscr{A})$. 则$\lambda\in\mathrm{spec}_{\mathbb{R}}(\mathscr{A}^*)$ 且关于$\mathscr{A}$和λ的特征子空间$V_{\mathscr{A}}^{\lambda}$等于关于$\mathscr{A}^*$和$\lambda$的特征子空间]

$0\in\mathrm{spec}_{\mathbb{R}}(\mathscr{A})\cap\mathrm{spec}_{\mathbb{R}}(\mathscr{A}^*)$.

即 $V_{\mathscr{A}}^0=V_{\mathscr{A}^*}^0$

$\Longrightarrow\ker(\mathscr{A}^*)=\ker(\mathscr{A})$.

法二:设$\vec{x}=x_1\vec{e_1}+\cdots+x_n\vec{e_n}$在$\ker(\mathscr{A})$中.

则 $A(x_1,\cdots x_n)^t=(0,\cdots,0)^t$

令 $(y_1,\cdots y_n)^t=A^t(x_1,\cdots,x_n)^t$

$\because AA^t=A^tA$

$\therefore y_1^2+\cdots+y_n^2=(y_1,\cdots,y_n)(y_1,\cdots y_n)^t$

$=(x_1,\cdots,x_n)AA^t(x_1,\cdots x_n)^t$

$=(x_1,\cdots x_n)A^tA(x_1,\cdots,x_n)^t=0$.

于是$y_1=y_2=\cdots=y_n=0\Longrightarrow x\in\ker(\mathscr{A}^*)$ 即 $\ker(\mathscr{A})\subset\ker(\mathscr{A}^*)$.

类似可证 $\ker(\mathscr{A}^*)\subset\ker(\mathscr{A})$.

法三：设$\mathscr{A}$在V的某组单位正交基$\vec{e_1},\cdots,\vec{e_n}$下的矩阵

$$A=\begin{bmatrix} N(\alpha_1,\beta_1) & & & & & \\ & \ddots & & & & \\ & & N(\alpha_s,\beta_s) & & & \\ & & & \lambda_{2s+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}.$$

其中$\lambda_{2s+1},\cdots,\lambda_t$非零. 而$\lambda_{t+1}=\cdots=\lambda_n=0$. $\because N(\alpha,\beta)$可逆.

所以 $A=\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow A^t=\begin{pmatrix} B^t & 0 \\ 0 & 0 \end{pmatrix}$.

其中$B\in GL_n(\mathbb{R})$. 由此可知，$\vec{x}\in\ker(\mathscr{A}) \Longleftrightarrow \vec{x}=x_{t+1}\vec{e}_{t+1}+\cdots+x_n\vec{e_n}$

$\Longleftrightarrow \vec{x}\in\ker(\mathscr{A}^*)$.

(ii). 设$\vec{x}\in im(\mathscr{A})$. 则$\exists\, \vec{y}\in V$ s.t $\vec{x}=\mathscr{A}(\vec{y})$.

设$\vec{z}$是$\ker(\mathscr{A})$中任意元. 计算

$$(\vec{z}\,|\,\vec{x})=(\vec{z}\,|\,\mathscr{A}(\vec{y}))=(\mathscr{A}^*(\vec{z})\,|\,\vec{y}).$$

由(i) $\ker(\mathscr{A})=\ker(\mathscr{A}^*) \Longrightarrow \mathscr{A}^*(\vec{z})=0$.

故$(\vec{z}\,|\,\vec{x})=0$. 即$\vec{z}\perp\vec{x}$.

从而 $\vec{x}\in\ker(\mathscr{A})^\perp$ 即 $im(\mathscr{A})\subset\ker(\mathscr{A})^\perp$.

$\because$ $\dim(im(\mathscr{A}))=\dim(V)-\dim(\ker(\mathscr{A}))$

$\dim(\ker(\mathscr{A})^\perp)=\dim(V)-\dim(\ker(\mathscr{A}))$.

于是 $im(\mathscr{A})\subset\ker(\mathscr{A})^\perp \Longrightarrow im(\mathscr{A})=\ker(\mathscr{A})^\perp$.

(iii) $\because\mathscr{A}$正规, $\therefore\mathscr{A}^*$正规.

由(ii) $\ker(\mathscr{A}^*)^\perp=im(\mathscr{A}^*)$

$\because$ $\ker(\mathscr{A})=\ker(\mathscr{A}^*)$

$\therefore$ $\ker(\mathscr{A})^\perp=\ker(\mathscr{A}^*)^\perp$

$$\therefore\quad \begin{array}{ccc} \ker(\mathscr{A})^\perp & \overset{}{\underset{(ii)}{=}} & im(\mathscr{A}) \\ \| & & \| \\ \ker(\mathscr{A}^*)^\perp & = & im(\mathscr{A}^*) \end{array}$$

课程内容回顾

6. $\mathcal{A}$ 正定算子及其等价条件.

7. 斜对称算子

8. 正交算子.

eg. 设 $A \in O_n(\mathbb{R})$ 且 $-1 \notin spec_{\mathbb{R}}(A)$. 证明:

(i) $E+A$ 可逆

(ii) $S := (E-A)(E+A)^{-1}$ 斜对称

(iii) $A = (E-S)(E+S)^{-1}$.

证: 设 $P \in O_n(\mathbb{R})$ s.t

$$A = P^t \begin{bmatrix} N(\cos(\theta_1), \sin(\theta_1)) & & & & & \\ & \ddots & & & & \\ & & N(\cos(\theta_s), \sin(\theta_s)) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} P$$

$$E+A = P^t \begin{bmatrix} N(1+\cos(\theta_1), \sin(\theta_1)) & & & & & \\ & \ddots & & & & \\ & & N(1+\cos(\theta_s), \sin(\theta_s)) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} P$$

(i) 由计算得

(ii)
$$(E_2 + N(\cos(\theta), \sin(\theta)))^{-1} = N(1+\cos(\theta), \sin(\theta))^{-1}$$
$$= \frac{1}{2+2\cos(\theta)} N(1+\cos(\theta), -\sin\theta)$$

且
$$B_2(\theta) := (E_2 - N(\cos(\theta), \sin(\theta)))(E_2 + N(\cos(\theta), \sin(\theta)))^{-1}$$
$$= \frac{1}{2+2\cos(\theta)} \begin{pmatrix} 0 & 2\sin(\theta) \\ -2\sin(\theta) & 0 \end{pmatrix} \in SSM_2(\mathbb{R}).$$

由矩阵分块运算得

$$(E-A)^{-1}(E+A)=P^t\begin{bmatrix} B_2(\theta_1) & & & & & \\ & \ddots & & & & \\ & & B_2(\theta_3) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}P \quad \in SSM_n(\mathbb{R}).$$

法二：令 $C=(E-A)(E+A)^{-1}$. 则

$$C^t=(E+A^t)^{-1}(E-A^t)=(E+A^{-1})^{-1}(E-A^{-1})$$
$$=(E+A^{-1})^{-1}A^{-1}A(E-A^{-1})=(A+E)^{-1}(A-E)$$
$$=(A-E)(A+E)^{-1}=-C.$$

(iii) 由课上例 7.2. $E+S$ 可逆.

由矩阵分块计算性质.

验证 $N(\cos(\theta),\sin(\theta))=(E_2-B_2(\theta))(E_2+B_2(\theta))^{-1}$

$$(E_2+B_2(\theta))^{-1}=N\left(\frac{1}{2}+\frac{1}{2}\cos(\theta),\ \frac{1}{2}\sin(\theta)\right).$$

于是 $(E_2-B_2(\theta))(E_2+B_2(\theta))^{-1}=N\left(1,\ \dfrac{\sin(\theta)}{1+\cos(\theta)}\right)N\left(\frac{1}{2}+\frac{1}{2}\cos(\theta),\ \frac{1}{2}\sin(\theta)\right)$

$$=N(\cos(\theta),\sin(\theta)).$$

□.

9. 谱分解定理

平方根定理

极化分解