第七次习题课

一、线性映射

1. $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ 线性映射与子空间

2. φ单 $\Longleftrightarrow$ $ker(\varphi)=\{\vec{0}_n\}$

hw1. 设φ是$\mathbb{R}^n \to \mathbb{R}^m$是线性映射且双射，$\varphi^{-1}$是$\varphi$的逆映射.

证：(1) φ^{-1}是线性映射；

(2) $n=m$.

证明：(1) $\varphi^{-1}: \mathbb{R}^m \to \mathbb{R}^n$

$\forall \vec{y}_1, \vec{y}_2 \in \mathbb{R}^m$, $\forall \alpha, \beta \in \mathbb{R}$ $\exists!\ x_1, x_2 \in \mathbb{R}^n$

s.t. $\varphi(\vec{x}_1)=\vec{y}_1$, $\varphi(\vec{x}_2)=\vec{y}_2$.

$\therefore \vec{x}_1=\varphi^{-1}(\vec{y}_1)$, $\vec{x}_2=\varphi^{-1}(\vec{y}_2)$

$\because \varphi(\alpha\vec{x}_1+\beta\vec{x}_2)=\alpha\varphi(\vec{x}_1)+\beta\varphi(\vec{x}_2)=\alpha\vec{y}_1+\beta\vec{y}_2$

$\therefore \varphi^{-1}(\alpha\vec{y}_1+\beta\vec{y}_2)=\alpha\vec{x}_1+\beta\vec{x}_2$

又 $\alpha\vec{x}_1+\beta\vec{x}_2=\alpha\varphi^{-1}(\vec{y}_1)+\beta\varphi^{-1}(\vec{y}_2)$

$\therefore \varphi^{-1}(\alpha\vec{y}_1+\beta\vec{y}_2)=\alpha\varphi^{-1}(\vec{y}_1)+\beta\varphi^{-1}(\vec{y}_2)$

$\therefore$ φ^{-1}是线性映射.

(2). 由对偶定理可知：$dim(ker(\varphi))+dim(im(\varphi))=n$.

φ是单射 $\Longrightarrow ker\varphi=\{\vec{0}_n\} \Longrightarrow dim(ker(\varphi))=0$.

φ是满射 $\Longrightarrow im\varphi=\mathbb{R}^m \Longrightarrow dim(im(\varphi))=m$

因此 $m=n$.

注：$dim(ker(\varphi))+dim(im(\varphi))=n$ $\varphi: \mathbb{R}^n \to \mathbb{R}^m$

① φ是单射，则$n \le m$

φ是单射 $\Longleftrightarrow ker(\varphi)=\{\vec{0}_n\} \Longleftrightarrow dim(ker(\varphi))=0$

$\Longleftrightarrow dim(im(\varphi))=n$

$im(\varphi) \subseteq \mathbb{R}^m \Longrightarrow n \le m$.

② φ 是满射，则 $n \geq m$.

φ 是满射 $\Longleftrightarrow$ $im(\varphi) = \mathbb{R}^m$ $\Longleftrightarrow$ $\dim(im(\phi)) = m$

3. $\phi: \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\sum_{j=1}^{n} a_j \vec{v_j} \longmapsto \sum_{j=1}^{n} a_j \vec{w_j} \qquad a_1, \cdots, a_n \in \mathbb{R}.$$

若 $A \in \mathbb{R}^{m \times n}$, $\phi_A: \mathbb{R}^n \to \mathbb{R}^m$ 满足 $\phi_A(\vec{e_j}) = \vec{A}^{(j)}$

hw 3. $\varphi: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \end{pmatrix}$$

证: (1) φ 是线性映射:

$$\forall \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3, \quad \alpha, \beta \in \mathbb{R}$$

验证 $\varphi(\alpha\vec{x} + \beta\vec{y}) = \alpha\varphi(\vec{x}) + \beta(\vec{y})$ 即可.

(2). 由定义

$A = (\varphi(\vec{e_1}), \varphi(\vec{e_2}), \varphi(\vec{e_3}))$ 计算出

$$\varphi(\vec{e_1}) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \varphi(\vec{e_2}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \varphi(\vec{e_3}) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

故 $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

回顾: 旋转 $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$

$$R_\theta(\vec{e_1}, \vec{e_2}) = (\vec{e_1}, \vec{e_2}) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\varphi(\vec{e_1},\vec{e_2},\vec{e_3}) = (\vec{e_1},\vec{e_2},\vec{e_3},\vec{e_4})\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\underset{\mathbb{R}^3}{\uparrow} \qquad\qquad \underset{\mathbb{R}^4}{\uparrow}$$

(3). 利用初等行变换得

$$A \to \cdots \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A)=2 \quad \therefore \dim(\text{im}\varphi)=2.$

由对偶定理得：$\dim(\ker\varphi)=3-2=1.$

注：此时是三个未知量.

(4) $\ker(\varphi)$ 对应的齐次线性方程组是

$$\begin{cases} x_1+x_2=0 \\ x_2+x_3=0 \end{cases} \iff \begin{cases} x_1=-x_2 \\ x_3=-x_2 \end{cases}$$

于是，$\ker(\varphi)$ 的一组基是 $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$\because \text{im}(\varphi)=V_c(A)$ 且 $\dim(\text{im}\varphi)=2.$

$\therefore$ A中任意两个线性无关的列向量是 $\text{im}(\varphi)$ 的一组基.

例：$\text{im}(\varphi)$ 的一组基是 $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

4. $\ker(\varphi_A)=\text{sol}(H)$ 且 L 相容 $\iff \vec{b}\in \text{im}(\varphi).$

5. a. $\varphi: \mathbb{R}^n \to \mathbb{R}^m$

$$\dim(\ker(\varphi))+\dim(\text{im}(\varphi))=n$$

6. $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ 　φ单 $\Longleftrightarrow$ φ满

hw2、设 $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ 线性映射，且 $\forall \vec{x} \in \mathbb{R}^n$，$\varphi(\vec{x}) \in \langle \vec{x} \rangle$.

证明：$\exists \lambda \in \mathbb{R}$ s.t. $\forall \vec{x} \in \mathbb{R}^n$ $\varphi(\vec{x}) = \lambda \vec{x}$.

证：设 $\vec{v_1}, \cdots, \vec{v_n}$ 是 $\mathbb{R}^n$ 的一组基，且 $\varphi(\vec{v_i}) = \lambda_i \vec{v_i}$，$\lambda_i \in \mathbb{R}$

$i = 1, 2, \cdots, n$.

下证 $\lambda_1 = \lambda_2 = \cdots = \lambda_n$

取 $\vec{x} = \vec{v_1} + \cdots + \vec{v_n}$，则 $\exists \lambda \in \mathbb{R}$ s.t. $\varphi(\vec{x}) = \lambda \vec{x}$

则 $\varphi(\vec{v_1} + \cdots + \vec{v_n}) = \varphi(\vec{v_1}) + \cdots + \varphi(\vec{v_n}) = \lambda_1 \vec{v_1} + \cdots + \lambda_n \vec{v_n}$

$= \lambda(\vec{v_1} + \cdots + \vec{v_n})$

$\Longrightarrow (\lambda_1 - \lambda)\vec{v_1} + \cdots + (\lambda_n - \lambda)\vec{v_n} = \vec{0}$

$\because$ $\vec{v_1}, \cdots, \vec{v_n}$ 线性无关 $\therefore$ $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$

$\forall \vec{y} \in \mathbb{R}^n$ $\exists!$ $a_1, \cdots, a_n \in \mathbb{R}$ s.t. $\vec{y} = a_1 \vec{v_1} + \cdots + a_n \vec{v_n}$

$$\begin{aligned}\Longrightarrow \varphi(\vec{y}) &= \varphi(a_1\vec{v_1} + \cdots + a_n\vec{v_n}) \\ &= a_1\varphi(\vec{v_1}) + \cdots + a_n\varphi(\vec{v_n}) \\ &= \lambda(a_1\vec{v_1} + \cdots + a_n\vec{v_n}) \\ &= \lambda\vec{y}.\end{aligned}$$

$\square$

hw5. 证："$\Leftarrow$"

$$\varphi(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}, \quad \varphi(\vec{y}) = \begin{pmatrix} f_1(\vec{y}) \\ \vdots \\ f_m(\vec{y}) \end{pmatrix}$$

$$a\varphi(\vec{x}) + \beta\varphi(\vec{y}) = \begin{pmatrix} af_1(\vec{x}) + \beta f_1(\vec{y}) \\ \vdots \\ af_m(\vec{x}) + \beta f_m(\vec{y}) \end{pmatrix} = \begin{pmatrix} f_1(a\vec{x} + \beta\vec{y}) \\ \vdots \\ f_m(a\vec{x} + \beta\vec{y}) \end{pmatrix}$$

$$= \varphi(a\vec{x} + \beta\vec{y})$$

$\therefore$ φ是线性映射。

"$\Longrightarrow$" $\alpha\varphi(\vec{x})+\beta\varphi(\vec{y})=\varphi(\alpha\vec{x}+\beta\vec{y})$

$\therefore$ $\alpha f_i(\vec{x})+\beta f_i(\vec{y})=f_i(\alpha\vec{x}+\beta\vec{y})$.

$f_1,\cdots,f_m$为线性函数

二. 矩阵的运算

6.1 线性映射在标准基下的矩阵表示

$$\varphi(\vec{e_j})=\sum_{i=1}^{m}a_{ij}\vec{\varepsilon_i}$$

$$A=(\varphi(\vec{e_1}),\cdots,\varphi(\vec{e_n}))=(a_{ij})_{m\times n}$$

$$\Phi:\ Hom(\mathbb{R}^n,\mathbb{R}^m)\longrightarrow\mathbb{R}^{m\times n}\qquad \Psi:\ \mathbb{R}^{m\times n}\longrightarrow Hom(\mathbb{R}^n,\mathbb{R}^m)$$

$$\phi\longmapsto A_\phi.\qquad A_\phi\longmapsto\phi_A$$

$$\Psi\circ\Phi=id_{Hom(\mathbb{R}^n,\mathbb{R}^m)}\qquad \Phi\circ\Psi=id_{\mathbb{R}^{m\times n}}$$

a. $im(\phi)=V_c(A)$, $\dim(im(\phi))=rank(A)$.

ϕ满射 $\Longleftrightarrow$ A行满秩.

b. $\dim(\ker(\phi))=n-rank(A)$ ϕ单射 $\Longleftrightarrow$ A列满秩

c. ϕ双射 $\Longleftrightarrow$ $m=n$ A满秩

6.2 线性映射的运算

映射的加法与数乘

6.3 矩阵的线性运算

推论: $Hom(\mathbb{R}^n,\mathbb{R}^m)=\{\varphi\mid\varphi:\mathbb{R}^n\to\mathbb{R}^m\}$.

$$\Phi:\ Hom(\mathbb{R}^n,\mathbb{R}^m)\longrightarrow\mathbb{R}^{m\times n}$$

$$\phi\longmapsto A_\phi$$

$$\Psi:\ \mathbb{R}^{m\times n}\longrightarrow Hom(\mathbb{R}^n,\mathbb{R}^m)$$

$$A\longmapsto\phi_A$$

$$\Phi(\lambda\phi+\mu\psi)=\lambda\Phi(\phi)+\mu\Phi(\psi)$$

$$\Psi(\lambda A+\mu B)=\lambda\Psi(A)+\mu\Psi(B)$$

6.4 矩阵的乘法

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\psi"] \arrow[dr, "\phi\circ\psi"'] & \mathbb{R}^s \arrow[d, "\phi"] \\ & \mathbb{R}^m \end{tikzcd}$$

1. 矩阵的乘法不满足交换律，满足结合律
2. $(AB)^t = B^tA^t$
3. $\mathrm{rank}(A) = \mathrm{rank}(BA) = \mathrm{rank}(AC)$

   $A \in \mathbb{R}^{m\times n}$，B是m阶满秩，C是n阶满秩.

7.1 $M_n(\mathbb{R})$ 上的运算

$A^t = A$： A为对称矩阵

$A^t = -A$： A为斜对称矩阵

$A^2 = A$： A为幂等矩阵

$A^k = 0$： A为幂零矩阵

7.2. 交换不变量与中心元

$tr(AB) = tr(BA)$.

中心元：数乘矩阵

搬运工引理

eg1. 设矩阵 $A = (a_{ij}) \in \mathbb{R}^{m\times n}$，其中 $i = 1, 2, \cdots, m$，$j = 1, 2, \cdots, n$

令

$$J_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m} \qquad J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}$$

求矩阵乘法 J_mA 和 AJ_n.

解: $A=(a_{ij})_{m\times n}$

$$J_mA=\begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad AJ_n=\begin{pmatrix} 0 & a_{11} & a_{12} & \cdots & a_{1,n-1} \\ 0 & a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m1} & a_{m2} & \cdots & a_{m,n-1} \end{pmatrix}$$

搬运工原理:

$$J_mA=(E_{12}^{(m)}+E_{23}^{(m)}+\cdots+E_{m-1,m}^{(m)})A$$
$$=E_{12}^{(m)}A+E_{23}^{(m)}+\cdots+E_{m-1,m}^{(m)}A$$
$$=\begin{pmatrix}\vec{A_2}\\ \vec{0}\\ \vdots\\ \vec{0}\\ \vec{0}\end{pmatrix}+\begin{pmatrix}\vec{0}\\ \vec{A_3}\\ \vdots\\ \vec{0}\\ \vec{0}\end{pmatrix}+\cdots+\begin{pmatrix}\vec{0}\\ \vec{0}\\ \vdots\\ \vec{A_m}\\ \vec{0}\end{pmatrix}=\begin{pmatrix}\vec{A_2}\\ \vec{A_3}\\ \vdots\\ \vec{A_m}\\ \vec{0}\end{pmatrix}$$

$$AJ_n=A(E_{12}^{(n)}+E_{23}^{(n)}+\cdots+E_{n-1,n}^{(n)})$$
$$=AE_{12}^{(n)}+AE_{23}^{(n)}+\cdots+AE_{n-1,n}^{(n)}$$
$$=(\vec{0},\vec{A}^{(1)},\vec{0},\cdots,\vec{0},\vec{0})+(\vec{0},\vec{0},\vec{A}^{(2)},\cdots\vec{0},\vec{0})+\cdots+(\vec{0},\cdots\vec{0},\vec{A}^{(m-1)})$$
$$=(\vec{0},\vec{A}^{(1)},\vec{A}^{(2)},\cdots,\vec{A}^{(n-2)},\vec{A}^{(n-1)})$$

eg2. 设 $a.b.c\in\mathbb{R}$, $m\in\mathbb{Z}^+$, 证明

$$\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}^m=\begin{pmatrix}1&ma&\frac{m(m-1)}{2}ab+mc\\0&1&mb\\0&0&1\end{pmatrix}$$

证: 一、可以用数学归纳法.

二. $A=\begin{pmatrix}0&a&c\\0&0&b\\0&0&0\end{pmatrix}$, 则 $\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}^m=(E_3+A)^m$

$$=\sum_{k=0}^{m}\binom{m}{k}E_3^{m-k}A^k$$

注: $AE_3=E_3A=A$.

$A^0=E_3$, $A^1=A$, $A^2=\begin{pmatrix}0&a&c\\0&0&b\\0&0&0\end{pmatrix}\begin{pmatrix}0&a&c\\0&0&b\\0&0&0\end{pmatrix}$

$$=\begin{pmatrix}0&0&ab\\0&0&0\\0&0&0\end{pmatrix}$$

$\cdots$ $A^k=O_3 \quad (k\geq 3)$.

$\therefore (E_3+A)^m=E_3+\binom{m}{1}A+\binom{m}{2}A^2$

$$=\begin{pmatrix}1&ma&\frac{m(m-1)}{2}ab+mc\\0&1&mb\\0&0&1\end{pmatrix}$$

eg3. $\varphi:\mathbb{R}^n\to\mathbb{R}^m$, $\psi:\mathbb{R}^s\to\mathbb{R}^n$, $\xi:\mathbb{R}^t\to\mathbb{R}^s$

分别对应在标准基下的矩阵 A, B, C.

$\operatorname{rank}(ABC)=\dim(\operatorname{Im}(\varphi\circ\psi\circ\xi))=\dim(\operatorname{Im}(\varphi)/\operatorname{Im}(\psi\circ\xi))$

$=\dim(\operatorname{Im}(\psi\circ\xi))-\dim(\ker(\varphi)/\operatorname{Im}(\psi\circ\xi))$

$\geq \operatorname{rank}(BC)-\dim(\ker(\varphi)/\operatorname{Im}(\psi))$

$=\operatorname{rank}(BC)-\dim(\operatorname{Im}(\psi))+\dim(\operatorname{Im}(\varphi)/\operatorname{Im}(\psi))$

$=\operatorname{rank}(BC)-\operatorname{rank}(B)+\underset{\dim(\operatorname{Im}(\varphi\circ\psi))}{\underset{\parallel}{\operatorname{rank}(AB)}}$