Recall: • $\mathscr{A}\in L(V)$, $v\in V$. 若 $V=F[\mathscr{A}]v$, 则称 $\mathscr{A}$ 是 V 上循环算子. v 为 V 中循环向量, V 是关于 $\mathscr{A}$ 和 V 的循环空间.
简称 $\mathscr{A}$循环空间

• $\mathscr{A}\in L(V)$, V 为 $\mathscr{A}$循环空间 $\Leftrightarrow \mu_A=\chi_A$.

1、 $\mathscr{A}\in L(\mathbb{R}^3)$

$$A=\begin{pmatrix}1&1&0\\0&-1&1\\1&0&0\end{pmatrix},\quad v=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\mathbb{R}(\mathscr{A})\cdot v$ 的基?

$v, \mathscr{A}v, \mathscr{A}^2v, \cdots$

$$\mathscr{A}\cdot v=\begin{pmatrix}1&1&0\\0&-1&1\\1&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}\qquad \mathscr{A}^2v=\mathscr{A}(\mathscr{A}v)=\begin{pmatrix}1&1&0\\0&-1&1\\1&0&0\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$$

$\det\begin{pmatrix}1&2&2\\1&0&1\\1&1&2\end{pmatrix}\neq 0$, $\Rightarrow$ $v, \mathscr{A}v, \mathscr{A}^2v$ 线性无关的.
$\Rightarrow$ 由于维数原因, $v, \mathscr{A}v, \mathscr{A}^2v, \mathscr{A}^3v$ 线性相关.

$v, \mathscr{A}v, \mathscr{A}^2v$ 为 $\mathbb{R}[\mathscr{A}]\cdot v$ 的一组基.

2. $A=\begin{pmatrix}J_2&\\&J_3\end{pmatrix}$ $\quad J_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad J_3=\begin{pmatrix}0&1&\\&0&1\\&&0\end{pmatrix}$

$$\mu_A=\operatorname{lcm}(\mu_{J_2},\mu_{J_3})\qquad \mu_{J_2}=t^2,\ \mu_{J_3}=t^3$$
$$=t^3$$

$\deg \mu_A = 3 < 5 = \deg \chi_A$. $\mu_A \neq \chi_A$.

从而 F^5 不是 A-循环的. □

$\chi_A = t^5$.

3. $n=\dim V$, $\mathscr{A} \in \mathcal{L}(V)$, 若 $\mathscr{A}^0, \mathscr{A}^1, \cdots, \mathscr{A}^{n-1}$ 在 F 上线性无关

则 V 是 $\mathscr{A}$-循环的.

证: $\mathcal{L}(V)$ $\mathscr{A}+\mathscr{B}$. $k\cdot\mathscr{A}$

$\deg \mu_{\mathscr{A}} \le n$, 若 $\deg \mu_{\mathscr{A}} = m < n$

设 $\mu_{\mathscr{A}} = t^m + a_1 t^{m-1} + \cdots + a_m$

$\Rightarrow \mu_{\mathscr{A}}(\mathscr{A}) = 0$

$\mathscr{A}^m + a_1 \mathscr{A}^{m-1} \cdots + a_m \mathscr{A}^0 = 0$

$\Rightarrow \mathscr{A}^0, \cdots, \mathscr{A}^m$ 线性相关. 与题目矛盾

从而 $\deg \mu_{\mathscr{A}} = n = \deg \chi_{\mathscr{A}}$.

$\mu_{\mathscr{A}} = \chi_A$. 从而 V 是 $\mathscr{A}$-循环的. □

5. $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) \in M_n(F)$. 看作 F^n 上算子, 证: F^n 是 $\mathscr{A}$ 循环 $\Leftrightarrow$ λ_i, λ_j 两两不同.

证: $\mu_A = \operatorname{lcm}(t-\lambda_1, \cdots, t-\lambda_n)$

$\chi_A = (t-\lambda_1) \cdot \cdots \cdot (t-\lambda_n)$

F^n 是 $\mathscr{A}$ 循环 $\Leftrightarrow \mu_{\mathscr{A}} = \chi_A \Leftrightarrow \lambda_i, \lambda_j$ 两两不同. □

"$\Leftarrow$" 若 λ_i, λ_j 两两不同, 则 F^n 是 $\mathscr{A}$-循环.

取 $v = e_1 + \cdots + e_n$.

$\mathscr{A} \cdot v = \lambda_1 e_1 + \cdots + \lambda_n e_n$

$\mathcal{A}^i v = \lambda_1^i e_1 + \cdots + \lambda_n^i e_n$

说明 $v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v$ 构成 F^n 一组基即可.

$$(v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v) = (e_1, \cdots, e_n)\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

$\lambda_i\ \lambda_j$ 两两不同 $\Rightarrow (v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v)$ 是一组基.

"$\Rightarrow$" $\exists v$. s.t. $v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v$ 是一组基，而不知道 $\mathcal{A}$ 在这组基下矩阵.

$v = v_1 e_1 + \cdots + v_n e_n$

$\mathcal{A}^i v = v_1 \cdot \lambda_1^i e_1 + \cdots + v_n \cdot \lambda_n^i e_n$

$$\underbrace{(v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v)}_{\text{基}} = (e_1, \cdots, e_n)\underbrace{\begin{pmatrix} v_1 & v_1\lambda_1 & \cdots & v_1\lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ v_n & v_n\lambda_1 & & v_n\lambda_n^{n-1} \end{pmatrix}}_{B}$$

$\Rightarrow \det B \neq 0$

$$\det B = v_1 \cdots v_n \cdot \det\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

$\Rightarrow v_1, \cdots, v_n$ 全部不为0，且 $\lambda_i \neq \lambda_j\ \forall i, j$.

6、$\mathcal{A}, \mathcal{B} \in L(V)$，$\mathcal{A}$ 是循环算子，$\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$，证明：$\mathcal{B} \in F[\mathcal{A}]$

证：$\mathcal{A}$ 为循环算子 $\Rightarrow \exists v \in V$ s.t. $V = F[\mathcal{A}] \cdot v$. $n = \dim V$

$v, \mathcal{A}v, \cdots, \mathcal{A}^{n-1}v$ 为 V 的基.

只需考虑 $\mathcal{B}$ 在基 $v, \cdots, \mathcal{A}^{n-1}v$ 下的取值.

$\mathcal{B}(v) \in V \Rightarrow \exists f(t) \in F[t]$, $\mathcal{B}(v) = f(\mathcal{A}) \cdot v$.

$\mathcal{B}(\mathcal{A}^i v) = \mathcal{A}^i(\mathcal{B}v) = \mathcal{A}^i \cdot (f(\mathcal{A}) \cdot v) = f(\mathcal{A}) \cdot (\mathcal{A}^i \cdot v)$

$\Rightarrow$ 对 $\forall \mathcal{A}^i v$, $i \geq 0$, $\mathcal{B}(\mathcal{A}^i v) = f(\mathcal{A}) \cdot (\mathcal{A}^i v)$

$$\mathcal{B}=f(\mathcal{A}).$$ □

4. $A=\begin{pmatrix}2&0&0\\1&2&-1\\1&0&1\end{pmatrix}$，求 A^k.

解：
$$\chi_A=\det(tE-A)=\begin{vmatrix}\lambda-2&0&0\\-1&\lambda-2&1\\-1&0&\lambda-1\end{vmatrix}=(\lambda-2)^2(\lambda-1)$$

$$\lambda_1=2 \qquad \lambda_2=1$$

$$V^{\lambda_1}:\ (\lambda_1E-A)\cdot\vec{x}=0 \qquad \begin{pmatrix}0&0&0\\-1&0&1\\-1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \Longrightarrow V^{\lambda_1}=\left\langle \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_1},\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{v_2}\right\rangle$$

$$V^{\lambda_2}:\ (\lambda_2E-A)\cdot\vec{x}=0 \qquad V^{\lambda_2}=\left\langle \underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{v_3}\right\rangle$$

$$令\ P=\begin{array}{c}\begin{pmatrix}1&0&0\\0&1&1\\1&0&1\end{pmatrix}\\ \begin{matrix}\lambda_1&\lambda_1&\lambda_2\end{matrix}\end{array} \qquad A\cdot P=P\cdot\begin{pmatrix}\lambda_1&&\\&\lambda_1&\\&&\lambda_2\end{pmatrix}$$

$$A(v_1,v_2,v_3)=(v_1,v_2,v_3)\cdot\begin{pmatrix}\lambda_1&&\\&\lambda_1&\\&&\lambda_2\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}2&&\\&2&\\&&1\end{pmatrix}$$

$$A=P\begin{pmatrix}2&&\\&2&\\&&1\end{pmatrix}P^{-1},\quad A^k=P\begin{pmatrix}2&&\\&2&\\&&1\end{pmatrix}^kP^{-1}$$

$$[P:E_3]\xrightarrow{\text{行变换}}[E_3:P^{-1}] \qquad P^{-1}=\begin{pmatrix}1&0&0\\1&1&-1\\-1&0&1\end{pmatrix}$$

$$A^k=\begin{pmatrix}2^k&0&0\\2^k-1&2^k&1-2^k\\2^k-1&0&1\end{pmatrix}$$ □

谱分解定理.

Recall：投影算子：V，$V=V_1\oplus\cdots\oplus V_k$. $V_i\subset V$ 子空间.

$$\pi_i:\ V\longrightarrow V$$
$$v=v_1+\cdots+v_k\longmapsto v_i$$

$$\Longrightarrow \begin{cases} \pi_i^2=\pi_i & \text{幂等} \\ \pi_i\pi_j=0, \ i\neq j. & \text{正交.} \\ \varepsilon=\pi_1+\cdots+\pi_k. \end{cases}$$

完全正交幂等方组：$\sigma_1,\cdots,\sigma_n$，　$\sigma_i: V\longrightarrow V$

$$\begin{cases} \sigma_i^2=\sigma_i \\ \sigma_i\sigma_j=0, \ i\neq j \\ \varepsilon=\sigma_1+\cdots+\sigma_n \end{cases}$$

完全正交幂等方组 $\Rightarrow V=\operatorname{Im}\sigma_1\oplus\cdots\oplus\operatorname{Im}\sigma_i$.

σ_i 为 V 到 $\operatorname{Im}\sigma_i$ 的投影映射(算子).

定理：$\mathscr{A}\in L(V)$，若 $\mathscr{A}$ 可对角化，则

(i) 存在完全正交幂等方组 π_i，$i=1,\cdots,k$，s.t.

$\mathscr{A}=\lambda_1\pi_1+\cdots+\lambda_k\pi_k$，　λ_i,λ_j 两两不同.

(ii) (i)中 λ_i 与 π_i 是唯一的.

(iii) $\exists f_1,\cdots,f_k\in F[x]$，s.t. $f_i(\lambda_j)=\delta_{ij}$，$\pi_i=f_i(\mathscr{A})$.

证：(i) $\mathscr{A}$ 可对角化，其特征值为 $\lambda_1,\cdots,\lambda_k$，$V^{\lambda_i}$ 为对应的特征子空间.

i.e. $V=V^{\lambda_1}\oplus\cdots\oplus V^{\lambda_k}$

取 π_i 为 V 到 V^{λ_i} 的投影. $\Rightarrow\{\pi_i \mid i=1,\cdots,k\}$ 为完全正交幂等方组

$\forall v=v_1+\cdots+v_k$，$v_i\in V^{\lambda_i}$

$$\begin{aligned}\mathscr{A}v=\mathscr{A}(v_1)+\cdots+\mathscr{A}(v_k)&=\lambda_1v_1+\cdots+\lambda_kv_k\\&=\lambda_1\pi_1(v)+\cdots+\lambda_k\pi_k(v)\\&=(\lambda_1\pi_1+\cdots+\lambda_k\pi_k)(v).\end{aligned}$$

由于 v 任取. $\Rightarrow \mathscr{A}=\lambda_1\pi_1+\cdots+\lambda_k\pi_k$.

(ii) 即若 $\mathscr{A}=a_1\sigma_1+\cdots+a_m\sigma_m$，则 $m=k$，且 $\sigma_i=\pi_i$ (指标换序后)

$\begin{cases} \{\sigma_i\} \text{为完全正交幂等组.} \\ a_i \neq a_j, \ i\neq j. \end{cases}$

a) a_i 为 $\mathcal{A}$ 的特征值.

$$V = \operatorname{im}(\sigma_1) \oplus \cdots \oplus \operatorname{im}\sigma_m.$$

$0 \neq w_i \in \operatorname{im}(\sigma_i)$, $\mathcal{A}(w_i)$

$$= a_1\sigma_1(w_i) + \cdots + a_m\sigma_m(w_i)$$
$$= a_i\sigma_i(w_i) = a_i \cdot w_i$$

$\Rightarrow$ a_i 为 $\mathcal{A}$ 的特征值.

$$\{a_1, \cdots, a_m\} \subseteq \{\lambda_1, \cdots, \lambda_k\}.$$

b) $\{\lambda_1, \cdots, \lambda_k\} \subseteq \{a_1, \cdots, a_m\}$.

设 λ 为 $\mathcal{A}$ 的特征值. $w = w_1 + \cdots + w_m$, $\exists\, w_{i_0} \neq 0$. $w_i \in \operatorname{im}(\sigma_i)$.

s.t. $\mathcal{A}w = \lambda w = \underline{\lambda w_1 + \cdots + \lambda w_m}$

$$\mathcal{A}w = (a_1\sigma_1 + \cdots + a_m\sigma_m)(w)$$
$$= a_1\sigma_1(w) + \cdots + a_m\sigma_m(w)$$
$$= \underline{a_1 w_1 + \cdots + a_m w_m}.$$

$(\lambda - a_1)w_1 + \cdots + (\lambda - a_m)w_m = 0$. $(\lambda - a_i)\cdot w_i \in \operatorname{im}\sigma_i$.

$\Rightarrow \begin{cases} (\lambda - a_1)w_1 = 0 \\ \vdots \\ (\lambda - a_m)w_m = 0 \end{cases}$ $\Rightarrow \exists\, i_0 \ \ w_{i_0} \neq 0$

从而 $(\lambda - a_{i_0}) = 0$

即 $\{\lambda_1, \cdots, \lambda_k\} \subseteq \{a_1, \cdots, a_m\}$.

c) 由 (a), (b) 已知 $m = k$, 即 $\mathcal{A} = \lambda_1\pi_1 + \cdots + \lambda_k\pi_k$

$$\mathcal{A} = \lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k.$$

现证 $\sigma_i = \pi_i$.

注意到

$$V = \underset{\cup|}{V^{\lambda_1}} \oplus \cdots \oplus \underset{\cup|}{V^{\lambda_k}}$$
$$V = \operatorname{im}\sigma_1 \oplus \cdots \oplus \operatorname{im}\sigma_k$$

σ_i 为 V 到 $\operatorname{im}\sigma_i$ 的投影, 只需证明 $\operatorname{im}\sigma_i = V^{\lambda_i}$

对 $\forall\, 0 \neq v_i \in \operatorname{im}\sigma_i$, $\underline{\mathcal{A}(v_i)} = \lambda_1\sigma_1(v_i) + \cdots + \lambda_k\sigma_k(v_i)$

$$= \underline{\lambda_i v_i}$$

$\Rightarrow \operatorname{im}\sigma_i \subseteq V^{\lambda_i}$

设 $d_i = \dim V^{\lambda_i}$ ， $e_i = \dim(\operatorname{im}\sigma_i)$

$n = d_1 + \cdots + d_k$ $\quad (d_1 - e_1) + \cdots + (d_k - e_k) = 0$

$= e_1 + \cdots + e_k$ $\quad d_i \geq e_i$

$\Rightarrow d_i = e_i$ ， $i = 1, \cdots, k$.

从而 $\operatorname{im}\sigma_i = V^{\lambda_i}$，命题(ii)得证. $\square$

(iii) 要 $f_i \in F[t]$， $f_i(\lambda_j) = \delta_{ij}$， 且 $\pi_i = f_i(\mathscr{A})$.

令 $f_i(t) = \prod\limits_{j \neq i} \dfrac{(t-\lambda_j)}{(\lambda_i - \lambda_j)}$ ， $f_i(t) = \dfrac{t-\lambda_1}{\lambda_i - \lambda_1} \cdot \cdots \cdot \dfrac{t-\lambda_{i-1}}{\lambda_i - \lambda_{i-1}} \cdot \dfrac{t-\lambda_{i+1}}{\lambda_i - \lambda_{i+1}} \cdot \cdots \cdot \dfrac{t-\lambda_k}{\lambda_i - \lambda_k}$

$f_i(\lambda_j) = \delta_{ij}$.

$\mathscr{A} = \lambda_1 \pi_1 + \cdots + \lambda_k \pi_k$.

$\mathscr{A}^2 = (\lambda_1\pi_1 + \cdots + \lambda_k\pi_k)(\lambda_1\pi_1 + \cdots + \lambda_k\pi_k) = \lambda_1^2 \pi_1 + \cdots + \lambda_k^2 \pi_k$

$\vdots$

$\mathscr{A}^i = \lambda_1^i \pi_1 + \cdots + \lambda_k^i \pi_k$.

$\vdots$

$\forall f \in F[t]$， $f(\mathscr{A}) = f(\lambda_1) \cdot \pi_1 + \cdots + f(\lambda_k)\pi_k$.

特别地 $f_i(\mathscr{A}) = f_i(\lambda_1)\pi_1 + \cdots + f_i(\lambda_k)\pi_k$.

$= \pi_i$. $\square$