$$\chi_A = |\lambda E - A| = \begin{vmatrix} \lambda-2 & -6 & 15 \\ -1 & \lambda-1 & 5 \\ -1 & -2 & \lambda+6 \end{vmatrix} = (\lambda+1)^3$$，它只有一个特征值 $\lambda_1 = -1$

$$\lambda_1 E - A = \begin{pmatrix} -3 & -6 & 15 \\ -1 & -2 & 5 \\ -1 & -2 & 15 \end{pmatrix} \to \begin{pmatrix} -1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim V^{-1} = 3 - 1 = 2$

$\Rightarrow$ 特征值为 -1 的 Jordan 块的个数只有2个

$|3| = \underline{1} + \underline{2}$

$$\Rightarrow J = \begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ & -1 \end{matrix}} & \\ & \boxed{-1} \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix} \in M_3(\mathbb{C})$$

令 $\chi_n = (t-\lambda_1)^{d_1} \cdots (t-\lambda_k)^{d_k}$

$\lambda_1, \cdots, \lambda_n$ 两两不同

$R(\lambda_i, l) = \operatorname{rank}(\lambda_i E - A)^l$

则 $N(\lambda_i, l) = R(\lambda_i, l-1) + R(\lambda_i, l+1) - 2R(\lambda_i, l)$

2. $V = \mathbb{R}[x]^{(n+1)} = \{f \in \mathbb{R}[x] \mid \deg(f) \le n\}$. 验证

$$(f|g) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right)$$

定义了 V 上一个内积. 计算 $\|x\|$.

pf: 回顾内积定义. $f(x,y)$ $\quad f: V \times V \to \mathbb{R}$, $\vec{x}, \vec{y} \mapsto f(\vec{x}, \vec{y})$. 是 V 上的双线性型 满足 $f(\vec{x}, \vec{x})$ 是正定的,则称 (V, f) 是一个欧氏空间, f 是 V 上的内积

$\forall \alpha, \beta \in \mathbb{R}$, $f, g, h \in V$, 验证

$(\alpha f + \beta g | h) = \alpha(f|h) + \beta(g|h)$, $(f | \alpha g + \beta h) = \alpha(f|g) + \beta(f|h)$.

双线性 ①

验证 $(f|g) = (g|f)$ 对称性 ②

验证正定性: ③ $\quad (f|f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)^2 \ge 0$. $\quad f\left(\frac{k}{n}\right) \in \mathbb{R}$ $\quad \Rightarrow (f|f)$ 是半正定

若 $(f|f)=0$，则 $f=0$

若 $(f|f)=0$，则 $\sum_{k=0}^{n} f\left(\frac{k}{n}\right)^2=0$ $\Rightarrow$ $f\left(\frac{k}{n}\right)=0$，$k=0,1,\cdots,n$.

$\Rightarrow$ f 有 $n+1$ 个两两不同的根.

$\deg_t(f)\le n$.

$\Rightarrow$ $f=0$.

$\Rightarrow$ $(f|f)$ 是正定的

$$\boxed{\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}}$$

$$\|x\|=\sqrt{(x,x)}=\sqrt{\sum_{k=0}^{n}\left(\frac{k}{n}\right)^2}=\sqrt{\frac{\sum_{k=0}^{n}k^2}{n^2}}=\sqrt{\frac{\frac{n(n+1)(2n+1)}{6}}{n^2}}=\sqrt{\frac{(n+1)(2n+1)}{6n}}$$

3. $\vec{u},\vec{v}\in V$

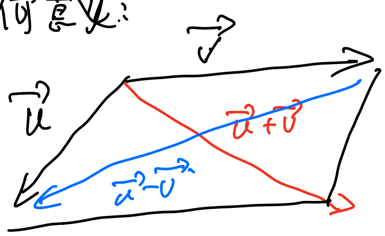
(a) $\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2\|\vec{u}\|^2+2\|\vec{v}\|^2$.

解：$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=(\vec{u}+\vec{v}|\vec{u}+\vec{v})+(\vec{u}-\vec{v}|\vec{u}-\vec{v})$

$$=(\vec{u}|\vec{u})+\cancel{2(\vec{u}|\vec{v})}+(\vec{v}|\vec{v})+(\vec{u}|\vec{u})-\cancel{2(\vec{u}|\vec{v})}+(\vec{v}|\vec{v})$$

$$=2\|\vec{u}\|^2+2\|\vec{v}\|^2.$$

几何意义：



2维欧氏空间中，平行四边形的四条边长的平方和等于两对角线平方之和.

(b) 若 $\|\vec{u}\|=\|\vec{v}\|$，则 $(\vec{u}+\vec{v})\perp(\vec{u}-\vec{v})$

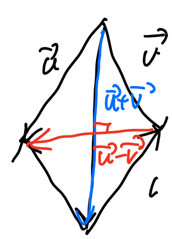
解：$(\vec{u}+\vec{v}|\vec{u}-\vec{v})=(\vec{u}|\vec{u})+\cancel{(\vec{u}|\vec{v})}-\cancel{(\vec{u}|\vec{v})}-(\vec{v}|\vec{v})$

$$=(\vec{u}|\vec{u})-(\vec{v}|\vec{v})$$

$$=\|\vec{u}\|^2-\|\vec{v}\|^2$$

$$=0$$

$\Rightarrow(\vec{u}+\vec{v})\perp(\vec{u}-\vec{v})$



几何意义：菱形的两对角线相互垂直.

4. 计算 $(J_{2m}(0))^2 \in M_{2m}(\mathbb{C})$ 的若当标准形.

$$J_{2m}(0)=\begin{pmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{pmatrix}. \qquad J_{2m}^2(0)=\begin{pmatrix}0&0&1&&\\&\ddots&\ddots&\ddots&\\&&\ddots&\ddots&1\\&&&\ddots&0\\&&&&0\end{pmatrix}$$

$2m-2$ $\neq 0$

$\mathrm{rank}(J_{2m}(0)) = 2m-1$ $\qquad$ $\mathrm{rank}(J_{2m}^2(0)) = 2m-2.$

$\boxed{\mathrm{rank}(J_{2m}^i(0)) = 2m-i-1.}$

$J_{2m}(0)$ 所有特征值均为0

$$\dim V^0 = 2m-(2m-2)=2.$$

$\Rightarrow$ $J_{2m}^2(0)$ 有两个若当块.

$2m = \square + \square$

$\mu_{J_{2m}(0)} = t^{2m}$ $\Rightarrow$ $\mu_{J_{2m}^2(0)} = t^m$.

$\boxed{J_{2m}^{2m}(0)=0 \Rightarrow (J_{2m}^2(0))^m=0 \Rightarrow \mu_{J_{2m}^2(0)} \mid t^m}$

故可设 $\mu_{J_{2m}^2(0)} = t^k$, $k\le m$.

若 $k<m$, 则 t^{2k} 能零化 $J_{2m}(0)$. 且 $2k<2m$. $\rightarrow \Leftarrow$

$\Rightarrow k=m$.

极小多项式次数代表若当块出现的最大阶数.

$\Rightarrow$ $J_{2m}^2(0)$ 中出现以0为特征值的若当块的阶数最大为 m.

$2m = m+m$

$\Rightarrow$ 若当标准形为 $\begin{pmatrix}J_m(0)&\\&J_m(0)\end{pmatrix}$.

(b). 计算 $(J_n(\lambda))^k \in M_n(\mathbb{C})$ 的若当标准形.

$$J_n(\lambda)=\begin{pmatrix}\lambda&1&&\\&\lambda&\ddots&\\&&\ddots&1\\&&&\lambda\end{pmatrix} \qquad (J_n(\lambda))^k = (\lambda E + J_n(0))^k$$

$$= \lambda^k E + \boxed{k\lambda^{k-1}J_n(0)} + \sum_{i=2}^{k} \binom{k}{i}\lambda^{k-i} J_n(0)^i$$

$$J_n(\lambda)^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & & \\ & \ddots & \ddots & \\ & & \ddots & k\lambda^{k-1} \\ & & & \lambda^k \end{pmatrix}$$

$\shortparallel$

A

$\Downarrow$

$J_n(\lambda)^k$ 的特征值全为 λ^k，$\chi_A = (t-\lambda^k)^n$

$$\mathrm{rank}(A-\lambda^k E) = \mathrm{rank}\begin{pmatrix} 0 & k\lambda^{k-1} & & \\ & \ddots & \ddots & \\ & & \ddots & k\lambda^{k-1} \\ & & & 0 \end{pmatrix} = n-1$$

$n-1$ $\neq 0$

$$\dim(V^{\lambda^k}) = n-(n-1) = 1$$

$\Rightarrow$ A 只有一个若当块，

$\Rightarrow J_A = J_n(\lambda^k)$

$$J_n(0) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

---

5. 设 $A \in \mathcal{L}(V)$ 为线性映射. 证明：若 A 有 k 维不变子空间，则 A 有 $n-k$ 维不变子空间.

解：设 U 是 k 维 A-不变子空间，不妨设 $0<k<n$. 设 $\vec{e}_1, \cdots, \vec{e}_k$ 是 U 的一组基，并把它扩充成 V 的一组基 $\vec{e}_1, \cdots, \vec{e}_k, \vec{e}_{k+1}, \cdots, \vec{e}_n$，则 A 在此基下矩阵为

$$A = \begin{pmatrix} B_{k\times k} & C_{k\times(n-k)} \\ O_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{pmatrix}.$$

$$A^t = \begin{pmatrix} B^t & O^t \\ C^t & D^t \end{pmatrix}$$

由于 $A \sim_s A^t$，故 $\exists\, P \in GL_n(F)$，s.t. $A^t = P^{-1}AP$，设

$$(\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_n) = (\vec{e}_1, \cdots, \vec{e}_n)P.$$

在基底 $\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_n$ 下的矩阵是

$$A^t = \begin{pmatrix} B^t & O^t \\ C^t & \boxed{D^t} \end{pmatrix},$$

$$A(\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_k, \vec{\varepsilon}_{k+1}, \cdots, \vec{\varepsilon}_n) = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_n)A^t$$

$A(\vec{\varepsilon}_{k+1},\cdots,\vec{\varepsilon}_n)=(\vec{\varepsilon}_{k+1},\cdots,\vec{\varepsilon}_n)D$

$\Rightarrow \langle \vec{\varepsilon}_{k+1},\cdots,\vec{\varepsilon}_n \rangle$ 是 $(n-k)$ 维 A-不变子空间.

V是实数域$\mathbb{R}$上的线性空间.
设$A\in \mathcal{L}(V)$,则V有一维或者二维的 A-子空间

解: $\mathbb{R}[t]$中的非平凡的不可约多项式的次数都不大于2.

$u_A\in\mathbb{R}[t]$. $\quad u_A=pq,\quad p,q\in\mathbb{R}[t]$. $0<\deg(p)\le 2$, p不可约.

$\deg(q)<\deg(u_A) \Rightarrow q(A)\ne \mathcal{O}$

$\Rightarrow \exists \vec{v}\in V$, s.t. $\vec{w}:=q(A)\vec{v}\ne\vec{0}$

设 $W=F[A]\cdot\vec{w}$, 则 $\dim(W)>0$.

$p(A)\vec{w}=p(A)q(A)\vec{v}=u_A(A)(\vec{v})=\vec{0}$

$\Rightarrow \deg(u_{A,\vec{w}})\le 2$

$f(A)\vec{w}$

$\Rightarrow \dim(W)=\deg(u_{A,\vec{w}})\le 2$. ✓.

注: 若存在一次因式p, $p=at+b$, $a\ne 0$, $\lambda=-\frac{b}{a}$是A的特征值 $\dim(V^{-\frac{b}{a}})=1$. $\Rightarrow$ 1维.

若一次因式全为2, 则存在一个二维不变子空间。

回顾行列式的几何意义

$W \subset V$, $V = W \oplus W^{\perp}$

$\forall \vec{x} \in V$, $\exists!\ x_1 \in W$, $x_2 \in W^{\perp}$,

s.t $\vec{x} = x_1 + x_2$

$x_1 = \pi_W(\vec{x}) = x_{\vec{W}}$, $x_2 = \vec{x}_{W^{\perp}}$

$d(\vec{x}, W) = \|\vec{x}_{W^{\perp}}\|$

$= \|x - \vec{x}_W\|$

$= \min_{y \in W} \{\|x - y\|\}$

定理 设 $W = \langle \vec{W_1}, ---, \vec{W_d} \rangle$ 且 $\vec{x} \in V$, 则

$$d^2(x, W) = \frac{\det(G(\vec{x}, \vec{W_1}, ---, \vec{W_d}))}{\det(G(\vec{W_1}, ----, \vec{W_d}))}$$

$\Downarrow$

设 $\mathbb{R}^n$ 中, $\vec{v_1}, \cdots, \vec{v_n}$ 线性无关, 由 $\vec{v_1}, \cdots, \vec{v_n}$ 构成的平行2n面体体积

$$P_n = |\det(v_1, ----, v_n)|.$$

$\mathbb{R}^n$ 中n个线性无关列向量构成的行列式可视为这些向量组成的平行多面体的"有向体积"

思考题 $A \in L(V)$, A是 $\mathcal{A}$ 在V某组基下的矩阵.

$$\det(\mathcal{A}) = \det(A)$$

问题: $\det(\mathcal{A})$ 几何意义 ?

$$\mathcal{A}(\underbrace{\vec{v_1}, ---, \vec{v_n}}_{一组基}) = (\vec{v_1}, \cdots, \vec{v_n})A$$

$$\det(\mathcal{A}\vec{v_1}, \cdots, \mathcal{A}\vec{v_n}) = \det(\vec{v_1}, ----, \vec{v_n})\det(A)$$

A可逆,

$$\Rightarrow \quad \det(A) = \frac{\det(\mathcal{A}\vec{v_1}, ----, \mathcal{A}\vec{v_n})}{\det(\vec{v_1}, ---- \vec{v_n})}$$

$\det\mathcal{A}$ 代表 映射后平行2n面体的体积与映射前平行2n面体的体积的比值.

长度与夹角：

例 $\mathbb{R}^4$ 中， $\vec{v_1}=\begin{pmatrix}1\\3\\2\\-1\end{pmatrix}$， $\vec{v_2}=\begin{pmatrix}-4\\2\\-3\\1\end{pmatrix}$

计算 $\|\vec{v_1}\|$，$\|\vec{v_2}\|$，求这两向量的夹角

解： $\|\vec{v_1}\|=\sqrt{(\vec{v_1}|\vec{v_1})}=\sqrt{1^2+3^2+2^2+(-1)^2}=\sqrt{15}$

$\|\vec{v_2}\|=\sqrt{4^2+2^2+3^2+1^2}=\sqrt{30}$.

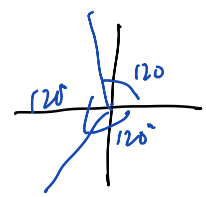
$(v_1|v_2)=1\times(-4)+3\times2+2\times(-3)+(-1)\times1=-4+6-6-1=-5$

$\underset{\substack{\uparrow\\[0,\pi]}}{\arccos}\left(\frac{(v_1|v_2)}{\|v_1\|\|v_2\|}\right)=\arccos\left(\frac{-5}{\sqrt{15}\sqrt{30}}\right)=\arccos\left(-\frac{\sqrt{2}}{6}\right)$

证明：在 n 维欧氏空间 V 中，两两夹角成钝角的元素不多于 $n+1$ 个.

Pf：对 n 作归纳. ① $n=1$ ✓

$n=2$. 由3 ✓



② 假设命题对 $n-1$ 维欧氏空间成立，下证 n 维成立.

反证：若在 V 中存在 $n+2$ 个元素 $\vec{v_1},\vec{v_2},\cdots,\vec{v}_{n+2}$，两两夹角成钝角

$(\alpha_i|\alpha_j)<0 \quad (i,j=1,2,\cdots,n+2,\ i\neq j)$

令 $W=\langle\vec{v_1}\rangle$，且当 V 分解为 $V=W\oplus W^{\perp}$.

$\Rightarrow \dim(W^{\perp})=n-1$

$\forall v_i\in V,\ i=2,3,\cdots,n+2,\ \exists k_i\in\mathbb{R},\ \vec{u_i}\in W^{\perp}$, s.t.

$\vec{v_i}=k_i\vec{v_1}+\vec{u_i},\ i=2,3,\cdots,n+2$

$\underset{\substack{\wedge\\0}}{(\vec{v_i}|\vec{v_1})}=(k_i\vec{v_1}+\vec{u_i}|\vec{v_1})=k_i\underset{\substack{\vee\\0}}{(\vec{v_1}|\vec{v_1})}+\underset{\substack{\parallel\\0}}{(\vec{u_i}|\vec{v_1})}$

$\Rightarrow k_i<0 \quad i=2,3,\cdots,n+2.$

$\forall i,j=2,3,\cdots,n+2$ 且 $i\neq j$，有

$0>(\vec{v_i}|\vec{v_j})=(k_i\vec{v_1}+\vec{u_i}|k_j\vec{v_1}+\vec{u_j})$

$$= (k_i k_j)(\vec{v_i}|\vec{v_1}) + (\vec{v_i}|\vec{v_j})$$

$$\Rightarrow (\vec{u_i}|\vec{u_j}) = -(k_i k_j)(\vec{v_1}|\vec{v_1}) + (\vec{v_i}|\vec{v_j}) < 0$$

$\vec{u_i} \in W^{\perp}$

$i, j = 2, \underbrace{\cdots, n+2}_{n+1}, \ i \neq j$

$\Rightarrow$ $n+1$ 维欧氏空间 $W^{\perp}$ 中存在 $n+1$ 个两两夹角为钝角的元素 $\times$

$\Rightarrow V$

Gram-Schmidt 正交化.

目标：对 n 维欧氏空间中 V 的任一组基 $\vec{v_1}, \cdots, \vec{v_n}$ 化为一组单位正交基.

$$\vec{\varepsilon_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|}$$

$$\vec{\varepsilon_2}' = \vec{v_2} - (\vec{v_2}|\vec{\varepsilon_1})\vec{\varepsilon_1}$$

$$\vec{\varepsilon_2} = \frac{\vec{\varepsilon_2}'}{\|\vec{\varepsilon_2}'\|}$$

$$\vdots$$

$$\boxed{\begin{aligned} \vec{\varepsilon_i}' &= \vec{v_i} - (\vec{v_i}|\vec{\varepsilon_1})\vec{\varepsilon_1} - \cdots - (\vec{v_i}|\vec{\varepsilon_{i-1}})\vec{\varepsilon_{i-1}} \\ \vec{\varepsilon_i} &= \frac{\vec{\varepsilon_i}'}{\|\vec{\varepsilon_i}'\|} \end{aligned}}$$

注：① 任一 n 维欧氏空间存在单位正交基.

② $W \subset V$，W 的一组单位正交基可扩充为 V 的一组单位正交基.

应用 问题：

设 $W = \langle \vec{w_1}, \cdots, \vec{w_d} \rangle \subset V$，其中 $0 < d < \dim(V)$，且 $\vec{w_1}, \cdots, \vec{w_d}$ 线性无关，给定 $\vec{v} \in V$，求 $\pi_W(\vec{v})$.

办法：求 W 的一组单位正交基 $\vec{\varepsilon_1}, \cdots, \vec{\varepsilon_d}$，则

$$\boxed{\pi_W(\vec{v}) = (\vec{v}|\vec{\varepsilon_1})\vec{\varepsilon_1} + \cdots + (\vec{v}|\vec{\varepsilon_d})\vec{\varepsilon_d}}$$

证明思路：把 $\vec{\varepsilon}_1,\cdots,\vec{\varepsilon}_d$ 扩充为 V 的一组单位正交基

$$\vec{\varepsilon}_1,\cdots,\vec{\varepsilon}_d,\vec{\varepsilon}_{d+1},\cdots,\vec{\varepsilon}_n$$

则 $\vec{\varepsilon}_{d+1},\cdots,\vec{\varepsilon}_n\in W^{\perp}$.

$\forall \vec{x}\in V$, $\exists\, \alpha_i\in\mathbb{R}$. $\vec{x}=\alpha_1\vec{\varepsilon}_1+\cdots+\alpha_n\vec{\varepsilon}_n$

$\alpha_i=(\vec{x}\,|\,\vec{\varepsilon}_i)$

$$\Rightarrow \vec{x}=\underbrace{(\vec{x}|\vec{\varepsilon}_1)\vec{\varepsilon}_1+\cdots+(\vec{x}|\vec{\varepsilon}_d)\vec{\varepsilon}_d}_{\in W,\ =\Pi_W(\vec{x})}+\underbrace{(\vec{x}|\vec{\varepsilon}_{d+1})\vec{\varepsilon}_{d+1}+\cdots+(\vec{x}|\vec{\varepsilon}_n)\vec{\varepsilon}_n}_{W^{\perp}}$$

例 设 $\vec{x}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\vec{w}_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\vec{w}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$.

计算 $\Pi_{\underbrace{\langle \vec{w}_1,\vec{w}_2\rangle}_{=W}}(\vec{x})$

解: $\vec{\varepsilon}_1=\dfrac{\vec{w}_1}{\|\vec{w}_1\|}=\dfrac{\vec{w}_1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}$

$\vec{\varepsilon}_2{}'=\vec{w}_2-(\vec{w}_2|\vec{\varepsilon}_2)\vec{\varepsilon}_1=\begin{pmatrix}0\\1\\1\end{pmatrix}-\dfrac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\end{pmatrix}$

$\vec{\varepsilon}_2=\dfrac{\vec{\varepsilon}_2{}'}{\|\vec{\varepsilon}_2{}'\|}=\dfrac{\vec{\varepsilon}_2{}'}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}=\dfrac{\sqrt{2}}{6}\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$\vec{\varepsilon}_1,\vec{\varepsilon}_2$ 为 W 的一组单位正交基.

$\Pi_W(\vec{x})=(\vec{x}|\vec{\varepsilon}_1)\vec{\varepsilon}_1+(\vec{x}|\vec{\varepsilon}_2)\vec{\varepsilon}_2$

$=\sqrt{2}\cdot\dfrac{\sqrt{2}}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}+\dfrac{2}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\dfrac{1}{\sqrt{6}}$

$=\begin{pmatrix}\frac{2}{3}\\ \frac{4}{3}\\ \frac{2}{3}\end{pmatrix}$

$d(\vec{x},W)=\|\vec{x}-\Pi_W(\vec{x})\|=\left\|\begin{pmatrix}\frac{1}{3}\\ -\frac{1}{3}\\ \frac{1}{3}\end{pmatrix}\right\|=\dfrac{\sqrt{3}}{3}$