1. Pf: (a) $\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=(\vec{u}+\vec{v}\mid\vec{u}+\vec{v})+(\vec{u}-\vec{v}\mid\vec{u}-\vec{v})=\|\vec{u}\|^2+\|\vec{v}\|^2+2(\vec{u}\mid\vec{v})+\|\vec{u}\|^2+\|\vec{v}\|^2-2(\vec{u}\mid\vec{v})$

$=2(\|\vec{u}\|^2+\|\vec{v}\|^2)$.

(b) $(\vec{u}+\vec{v}\mid\vec{u}-\vec{v})=\|\vec{u}\|^2-(\vec{u}\mid\vec{v})+(\vec{u}\mid\vec{v})-\|\vec{v}\|^2=0$

$\Rightarrow(\vec{u}+\vec{v})\perp(\vec{u}-\vec{v})$

(c) $(\vec{u}\mid\vec{v})$

$\|\vec{u}+\vec{v}\|^2=(\vec{u}+\vec{v}\mid\vec{u}+\vec{v})=\|\vec{u}\|^2+\|\vec{v}\|^2+2(\vec{u}\mid\vec{v})$ ①

$\|\vec{u}-\vec{v}\|^2=(\vec{u}-\vec{v}\mid\vec{u}-\vec{v})=\|\vec{u}\|^2+\|\vec{v}\|^2-2(\vec{u}\mid\vec{v})$. ②

①-②得$\Rightarrow$ $(\vec{u}\mid\vec{v})=\frac{1}{4}\|\vec{u}+\vec{v}\|^2-\frac{1}{4}\|\vec{u}-\vec{v}\|^2$.

2. (a). $(x\mid x)=\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$

$\Rightarrow\|x\|=\sqrt{\frac{1}{3}}=\frac{\sqrt{3}}{3}$

$\|x^2-1\|=\sqrt{(x^2-1\mid x^2-1)}=\sqrt{\int_0^1(x^2-1)^2dx}=\sqrt{\int_0^1(x^4-2x^2+1)\,dx}$

$=\sqrt{\frac{1}{5}x^5\Big|_0^1-\frac{2}{3}x^3\Big|_0^1+x\Big|_0^1}$

$=\sqrt{\frac{1}{5}-\frac{2}{3}+1}=\sqrt{\frac{8}{15}}=\frac{2\sqrt{30}}{15}$.

Def $\forall\vec{x}\in V$, $\sqrt{(x\mid x)}$ 是$\vec{x}$的长度, $\vec{y}\in V$, $\|\vec{x}-\vec{y}\|$为$\vec{x}$到$\vec{y}$之间的距离

V的一组基为 $\vec{v_1}=1,\vec{v_2}=x,\vec{v_3}=x^2$.

$1,\ 2\sqrt{3}x-\sqrt{3},$

$6\sqrt{5}\,(x^2-x+\frac{1}{6})$

3. $\begin{pmatrix}1&0&0&-1\\2&1&0&1\\-1&0&0&1\\1&1&0&2\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&-1\\0&1&0&3\\0&0&0&0\\0&1&0&3\end{pmatrix}\rightarrow\begin{pmatrix}1&0&0&-1\\0&1&0&3\\0&0&0&0\\0&0&0&0\end{pmatrix}$.

由 $\overset{A}{\text{rank}(A)}=2$ 可得, $W^{\perp}=\langle\vec{A_1}^t,\vec{A_2}^t\rangle$.

$\vec{\varepsilon_1}=\frac{\vec{A_1}^t}{\|\vec{A_1}\|}=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

$\vec{\varepsilon_2}'=\vec{A_2}^t-(\vec{A_2}^t\mid\vec{\varepsilon_1})\vec{\varepsilon_1}=\begin{pmatrix}2\\1\\0\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\1\\0\\\frac{3}{2}\end{pmatrix}$

$\vec{\varepsilon_2}=\frac{\vec{\varepsilon_2}'}{\|\vec{\varepsilon_2}'\|}=\frac{\sqrt{22}}{22}\begin{pmatrix}3\\2\\0\\3\end{pmatrix}$.

Gram-schmidt 正交化.

假设 $\vec{v_1},\cdots,\vec{v_k}\in V$ 线性无关

$\Rightarrow\exists$ 两两正交单位向量 $\vec{\varepsilon_1},\cdots,\vec{\varepsilon_k}$, 使得

$\langle\vec{v_1},\cdots,\vec{v_i}\rangle=\langle\vec{\varepsilon_1},\cdots,\vec{\varepsilon_i}\rangle$

$\varepsilon_i'=\vec{v_i}-(\vec{v_i}\mid\varepsilon_1)\varepsilon_1-\cdots-(\vec{v_i}\mid\varepsilon_{i-1})\varepsilon_{i-1}$

$\varepsilon_i=\frac{\varepsilon_i'}{\|\varepsilon_i'\|}$

4. 证明: $U_2^{\perp}\subseteq U_1^{\perp}$.

(1) 设 $\vec{x}\in U_2^{\perp}$, 则$\vec{x}$与U_2中的任意向量正交. 因为 $U_1\subseteq U_2$, 所以 $\vec{x}$与U_1中的任意向量正交.

由此可得 $U_2^{\perp}\subseteq U_1^{\perp}$.

①

解: $(U_1\cap U_2)^\perp = U_1^\perp + U_2^\perp$

Claim: $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$.

$W_1 \subset W_1+W_2$, $W_1^\perp \supset (W_1+W_2)^\perp$. 同理, $W_2^\perp \supset (W_1+W_2)^\perp$. 故 $W_1^\perp\cap W_2^\perp \supset (W_1+W_2)^\perp$.

反之, 设 $\vec{y}\in W_1^\perp\cap W_2^\perp$. 对 $\forall\ \vec{w}\in W_1+W_2$, $\exists\ \vec{w_1}\in W_1$, $\vec{w_2}\in W_2$, s.t $\vec{w}=\vec{w_1}+\vec{w_2}$.

$$\Rightarrow (\vec{y}|\vec{w}) = (\vec{y}|\vec{w_1}+\vec{w_2}) = (\vec{y}|\vec{w_1})+(\vec{y}|\vec{w_2}) = 0$$

$$\Rightarrow \vec{y}\in (W_1+W_2)^\perp$$

$$\Rightarrow W_1^\perp\cap W_2^\perp \subset (W_1+W_2)^\perp.$$

由 $W_1\cap W_2\subset W_1$ 可得, $(W_1\cap W_2)^\perp \supset W_1^\perp$, 同理 $(W_1\cap W_2)^\perp\supset W_2^\perp$.

故 $(W_1\cap W_2)^\perp \supset W_1^\perp + W_2^\perp$

$$\begin{aligned}
\dim(W_1^\perp+W_2^\perp) &= \dim(W_1^\perp)+\dim(W_2^\perp)-\dim(W_1^\perp\cap W_2^\perp)\\
&= \dim(W_1^\perp)+\dim(W_2^\perp)-\dim((W_1+W_2)^\perp)\\
&= \dim(V)-\dim(W_1)+\dim(V)-\dim(W_2)-(\dim(V)-\dim(W_1+W_2))\\
&= \dim(V)-(\dim(W_1)+\dim(W_2)-\dim(W_1+W_2))\\
&= \dim(V)-\dim(W_1\cap W_2)\\
&= \dim((W_1\cap W_2)^\perp).
\end{aligned}$$

法二: $(W^\perp)^\perp = W$.

$$(W_1^\perp+W_2^\perp)^\perp = (W_1^\perp)^\perp\cap(W_2^\perp)^\perp = W_1\cap W_2.$$

$$((W_1^\perp+W_2^\perp)^\perp)^\perp = (W_1\cap W_2)^\perp$$

$$\Rightarrow W_1^\perp+W_2^\perp = (W_1\cap W_2)^\perp.$$

5. 解: (a) $\forall\ A,B\in SO_n(\mathbb{R})$, 有 $\det(A)=\det(B)=1$, $A,B\in O_n(\mathbb{R})$

$$\det(AB^{-1}) = \det(A)\det(B^{-1}) = 1\cdot 1 = 1$$

$\Rightarrow SO_n(\mathbb{R})$ 是 $O_n(\mathbb{R})$ 的子群.

(b) 对 n 归纳. 当 $n=1$, $A=1$ 或 $A=-1$. 结论显然成立. 设 $n>1$ 且 $n-1$ 时结论成立.

因为 A 是上三角矩阵, 所以 $A=\begin{pmatrix} a & C\\ 0_{n-1\times 1} & B\end{pmatrix}$.

②

其中 $a\in\mathbb{R}$，$B\in M_{n-1}(\mathbb{R})$ 是上三角矩阵. $C\in\mathbb{R}^{1\times(n-1)}$. A 正交 ⇒ A 正规矩阵.

$$\Rightarrow A=\begin{pmatrix} a & O_{1\times(n-1)} \\ O_{(n-1)\times1} & B \end{pmatrix}$$

由命，

$$E=A^tA=\begin{pmatrix} a^2 & O_{1\times(n-1)} \\ O_{(n-1)\times1} & B^tB \end{pmatrix}$$

⇒ $a^2=1$ 且 B 正交. 由归纳假设可直接得出 A 是对角矩阵且在对角线上元素等于 ±1.

解: (a)⇒(b) $\vec{e_1},\cdots,\vec{e_n}$ 是 V 的一组单位正交基，则 $(\vec{e_i}|\vec{e_i})=1$，$(\vec{e_i}|\vec{e_j})=0$.

$\forall\vec{x},\vec{y}\in V$，$\vec{x}=\sum_{i=1}^n a_i\vec{e_i}$，$\vec{y}=\sum_{i=1}^n b_i\vec{e_i}$，这里 $a_i=(\vec{x}|\vec{e_i})$，$b_i=(\vec{y}|\vec{e_i})$.

$$(\vec{x},\vec{y})=\left(\sum_{i=1}^n a_i\vec{e_i}\,\Big|\,\sum_{i=1}^n b_i\vec{e_i}\right)=\sum_{i=1}^n (a_i|b_i)(\vec{e_i}|\vec{e_i})=\sum_{i=1}^n(\vec{x}|\vec{e_i})(\vec{y}|\vec{e_i})$$

(b)⇒(c) 令 $\vec{y}=\vec{x}$，$\|\vec{x}\|^2=\langle\vec{x}|\vec{x}\rangle=\sum_{i=1}^n(\vec{x}|\vec{e_i})^2$

(c)⇒(a). 对任意 $\vec{x}\in V$，$\|\vec{x}\|^2=\sum_{i=1}^n(\vec{x}|\vec{e_i})^2$ ⇒ $\vec{e_1},\cdots,\vec{e_n}$ 是 V 的一组单位正交基.

先设 $\vec{x}=\vec{e_1}$，则 $\|\vec{e_1}\|^2=\sum_{i=1}^n(e_1|e_i)^2=\|\vec{e_1}\|^4+\sum_{i=2}^n(\vec{e_1}|\vec{e_i})^2$

$$\Rightarrow \|\vec{e_1}\|^4\le\|\vec{e_1}\|^2 \Rightarrow \|\vec{e_1}\|^2\le1 \Rightarrow \|\vec{e_1}\|\le1$$

$\dim(V)=n$. ⇒ $\exists\,\vec{v}\in\langle\vec{e_2},\cdots,\vec{e_n}\rangle^\perp\setminus\{\vec{0}\}$.

$$\Rightarrow \|\vec{v}\|=\sum_{i=1}^n(\vec{v}|\vec{e_i})^2=(\vec{v}|\vec{e_1})^2\le\|v\|^2\|e_1\|^2 \Rightarrow \|\vec{e_1}\|\ge1.$$

⇒ $\vec{e_1}$ 是单位向量，

$$1=1+\sum_{i=2}^n(e_1|\vec{e_i})^2 \Rightarrow \sum_{i=2}^n(\vec{e_1}|\vec{e_i})=0 \Rightarrow (\vec{e_1}|\vec{e_i})=0,\ i=2,3,\cdots,n.$$

⇒ $\vec{e_1}$ 与 $\vec{e_2},\cdots,\vec{e_n}$ 都正交.

同理可证对 $\forall\, i\in\{2,3,\cdots,n\}$，$\vec{e_i}$ 是单位向量并与 $\vec{e_1},\cdots,\vec{e_{i-1}},\vec{e_{i+1}},\cdots,\vec{e_n}$ 都正交

设 $A \in O_n(\mathbb{R})$ 且 $-1 \notin \mathrm{spec}_{\mathbb{R}}(A)$，证明：

(i) $E+A$ 可逆

(ii) $S := (E-A)(E+A)^{-1}$ 斜对称

(iii) $A = (E-S)(E+S)^{-1}$

解：(i) 设 $P \in O_n(\mathbb{R})$, s.t.

$$\Rightarrow N(\cos(\theta_1), \sin(\theta_1)) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$A = P^t \begin{pmatrix} N(\cos(\theta_1), \sin(\theta_1)) & & & & \\ & \ddots & & & \\ & & N(\cos(\theta_s), \sin(\theta_s)) & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} P$$

$$E+A = P^t \begin{pmatrix} N(1+\cos(\theta_1), \sin(\theta_1)) & & & & \\ & \ddots & & & \\ & & N(1+\cos(\theta_s), \sin(\theta_s)) & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} P$$

$\Rightarrow$ $E+A$ 可逆.

(ii)
$$\begin{pmatrix} 1+\cos\theta & -\sin\theta \\ \sin\theta & 1+\cos\theta \end{pmatrix}^{-1} = \frac{\begin{pmatrix} 1+\cos\theta & \sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix}}{2+2\cos(\theta)} = \frac{N(1+\cos\theta, -\sin\theta)}{2+2\cos\theta}$$

$$\|$$

$$(E_2 + N(\cos\theta, \sin\theta))$$

且

$$B_2(\theta) = (E_2 - N(\cos\theta, \sin\theta))(E_2 + N(\cos\theta, \sin\theta))^{-1}$$
$$= \frac{\begin{pmatrix} 0 & 2\sin\theta \\ -2\sin\theta & 0 \end{pmatrix}}{2+2\cos(\theta)} \in SSM_2(\mathbb{R}).$$

$$\Rightarrow (E-A)^{-1}(E+A) = P^t \begin{pmatrix} B_2(\theta_1) & & & & \\ & \ddots & & & \\ & & B_2(\theta_s) & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} P \in SSM_n(\mathbb{R})$$

令 $C = (E-A)(E+A)^{-1}$

$$C^t = (E+A^t)^{-1}(E-A^t) = (E+A^{-1})^{-1}(E-A^{-1}).$$

$$= (E+A^{-1})^{-1}A^{-1}A(E-A^{-1}) = (A+E)^{-1}(A-E) = (A-E)(A+E)^{-1} = -C$$

结论：设 $M,N\in M_n(F)$ 且 M 可逆，如果 $MN=NM$，则 $NM^{-1}=M^{-1}N$.

$$MN=NM \Rightarrow M^{-1}MNM^{-1}=M^{-1}NM$$

$$\Rightarrow NM^{-1}=M^{-1}NM$$

$$\Rightarrow NM^{-1}=M^{-1}N\cancel{M}\cdot\cancel{M^{-1}}$$

$$\Rightarrow NM^{-1}=M^{-1}N.$$

(iii) $\underset{\downarrow}{E+S\text{可逆}}$，由矩阵分块计算性质，只需验证 $N(\cos\theta,\sin\theta)=(E_2-B_2(\theta))(E_2+B_2(\theta))^{-1}$

$S\in SSM_n(\mathbb{R})\Rightarrow \exists P\in O_n(\mathbb{R})$, s.t. $P^tAP=M$.

$$\Rightarrow P^t(E+A)P=E+M=\begin{pmatrix} N(1,\beta_1) & & & & & \\ & \ddots & & & & \\ & & N(1,\beta_s) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

$$\det(E+M)=(1+\beta_1^2)\cdots(1+\beta_s)^2\neq 0.$$

$\Rightarrow E+M$ 可逆.

$$(E_2+B_2(\theta))^{-1}=N\left(\tfrac{1}{2}+\tfrac{1}{2}\cos\theta,\tfrac{1}{2}\sin\theta\right)$$

$$(E_2-B_2(\theta))(E_2+B_2(\theta))^{-1}=N\left(1,\frac{\sin\theta}{1+\cos\theta}\right)N\left(\tfrac{1}{2}+\tfrac{1}{2}\cos\theta,\tfrac{1}{2}\sin\theta\right)$$

$$=N(\cos\theta,\sin\theta)$$

$A\in M_n(\mathbb{R})$

① 正规，$A\sim_o B=\begin{pmatrix} N(\alpha_1,\beta_1) & & & & & \\ & \ddots & & & & \\ & & N(\alpha_s,\beta_s) & & & \\ & & & \lambda_{s+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$

② 实对称，$A\sim_o\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_s & & & \\ & & & \lambda_{s+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$