1.
$$\begin{cases} x_1-x_2-x_3+x_4=0 \\ x_1-x_2+x_3-3x_4=0 \end{cases} \quad H.$$

解：该方程组的系数矩阵

$$A=\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -3 \end{pmatrix}$$

通过初等行变换得，

$$A \to \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

于是，$\mathrm{rank}(A)=2$. 从而 $\dim(\mathrm{sol}(H))=4-2=2$. 由高斯消去法可得.

$$H \Leftrightarrow \begin{cases} x_1-x_2-x_3+x_4=0 \\ x_3-2x_4=0 \end{cases} \Leftrightarrow \begin{cases} x_1=x_2+x_4. \\ x_3=2x_4 \end{cases}$$

令 $x_2=1, x_4=0$，则 $x_1=1, x_3=0$，从而 $\vec{v_1}=(1,1,0,0)^t$ 是 H 的解.

令 $x_2=0, x_4=1$，则 $x_1=1, x_3=2$，从而 $\vec{v_2}=(1,0,2,1)^t$ 是 H 的解

易知 $\vec{v_1},\vec{v_2}$ 线性无关，故 $\mathrm{sol}(H)=\{\lambda_1\vec{v_1}+\lambda_2\vec{v_2} \mid \lambda_1,\lambda_2\in\mathbb{R}\}$.

2.
$$L \begin{cases} x_1+2x_2+5x_3=1 \\ x_1+3x_2-2x_3=1 \\ 3x_1+7x_2+8x_3=3 \end{cases}$$

解：该方程组的增广矩阵

$$B=(A|\vec{b})=\left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 1 & 3 & -2 & 1 \\ 3 & 7 & 8 & 3 \end{array}\right)$$

通过初等行变换，得

$$B \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & 1 & -7 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\Rightarrow \mathrm{rank}(B)=\mathrm{rank}(A)=2$，则 L 相容.

由高斯消去法可知，给定方程组等价于

$$\begin{cases} x_1+2x_2+5x_3=1 \\ x_2-7x_3=0 \end{cases} \Leftrightarrow \begin{cases} x_1=1-19x_3. \\ x_2=7x_3 \end{cases}$$

令 $x_3=0$，得 $\vec{v}=(1,0,0)^t$ 是 L 的一个特解.

以 A 为系数矩阵的齐次线性方程组 H 等价于

$$\begin{cases} x_1+2x_2+5x_3=0 \\ x_2-7x_3=0 \end{cases} \Leftrightarrow \begin{cases} x_1=-19x_3. \\ x_2=7x_3 \end{cases}$$

令 $x_3=1$，则 $\vec{w_1}=(-1,9,7,1)$ 是 H 的解空间一组基底，故

$Sol(L)=\vec{v}+\langle\vec{w_1}\rangle$

3. 在下列映射中，哪些是线性映射：（$\mathbb{R}^{1\times n}$ 上）

a) $\phi_1:[x_1,x_2,\cdots,x_n]\longmapsto[x_n,\cdots,x_2,x_1]$.　　ϕ

b) $\phi_2:[x_1,x_2,\cdots,x_n]\longmapsto[x_1,x_2^2,\cdots,x_n^n]$;

c) $\phi_3:[x_1,x_2,\cdots,x_n]\longmapsto[x_1,x_1+x_2,\cdots,x_1+x_2+\cdots+x_n]$.

解：$\forall[x_1,x_2,\cdots,x_n],[\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n]\in\mathbb{R}^{1\times n}$，$\alpha,\beta\in\mathbb{R}$，

$\phi_1(\alpha[x_1,x_2,\cdots,x_n]+\beta[\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n])=\phi_1([\alpha x_1+\beta\tilde{x}_1,\alpha x_2+\beta\tilde{x}_2,\cdots,\alpha x_n+\beta\tilde{x}_n])$

$=[\alpha x_n+\beta\tilde{x}_n,\cdots,\alpha x_2+\beta\tilde{x}_2,\alpha x_1+\beta\tilde{x}_1]$

$=\alpha[x_n,\cdots,x_2,x_1]+\beta[\tilde{x}_n,\cdots,\tilde{x}_2,\tilde{x}_1]$

$=\alpha\cdot\phi_1([x_1,x_2,\cdots,x_n])+\beta\cdot\phi_1([\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n])$.

$\Rightarrow$ ϕ_1 是线性映射.

b) 假设 ϕ_2 是线性映射，则 $2\phi_2([1,1,\cdots,1])=\phi_2(2[1,1,\cdots,1])=\phi_2([2,2,\cdots,2])$

$2\phi_2([1,1,\cdots,1])=2[1,1,\cdots,1]=[2,2,\cdots,2]$.

$\phi_2([2,2,\cdots,2])=[2,4,\cdots,2^n]\neq2\phi_2([1,1,\cdots,1])$.

$\Rightarrow$ ϕ_2 不是线性映射.

$\forall[x_1,\cdots,x_n],[\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n]\in\mathbb{R}^{1\times n}$，$\alpha,\beta\in\mathbb{R}$.

$\phi_3(\alpha[x_1,x_2,\cdots,x_n]+\beta[\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n])=\phi_3([\alpha x_1+\beta\tilde{x}_1,\alpha x_2+\beta\tilde{x}_2,\cdots,\alpha x_n+\beta\tilde{x}_n])$

$=[\alpha x_1+\beta\tilde{x}_1,\alpha x_1+\beta\tilde{x}_1+\alpha x_2+\beta\tilde{x}_2,\cdots,\sum_{i=1}^{n}(\alpha x_i+\beta\tilde{x}_i)]$

$=[\alpha x_1+\beta\tilde{x}_1,\alpha(x_1+x_2)+\beta(\tilde{x}_1+\tilde{x}_2),\cdots,\alpha\sum_{i=1}^{n}x_i+\beta\sum_{i=1}^{n}\tilde{x}_i]$

$=\alpha[x_1,x_1+x_2,\cdots,\sum_{i=1}^{n}x_i]+\beta[\tilde{x}_1,\tilde{x}_1+\tilde{x}_2,\cdots,\sum_{i=1}^{n}\tilde{x}_i]$

$=\alpha\phi_3([x_1,x_2,\cdots,x_n])+\beta\phi_3([\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_n])$.

$\Rightarrow\phi_3$ 是线性映射

4. 证明：设 W 的一组基为 $\vec{w_1},\cdots,\vec{w_d}$.

由于 ϕ 是满射，则 $\exists\ \vec{v_1},\cdots,\vec{v_d}\in\mathbb{R}^n$, s.t. $\phi(\vec{v_i})=\vec{w_i}$, $i=1,2,\cdots,d$.

由于 $\vec{w_1},\cdots,\vec{w_d}$ 线性无关，则 $\vec{v_1},\cdots,\vec{v_d}$ 线性无关.

由 $\vec{v_1},\cdots,\vec{v_d}\in\phi^{-1}(W)$ 得

$$\dim(\phi^{-1}(W))\geq d=\dim(W).$$

3. 证明：若 $a_0\neq0$，方阵

$$A=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & a_0\\ 1 & 0 & \cdots & 0 & 0 & a_1\\ 0 & 1 & \cdots & 0 & 0 & a_2\\ \vdots & & & & & \vdots\\ 0 & 0 & \cdots & 1 & 0 & a_{n-2}\\ 0 & 0 & \cdots & 0 & 1 & a_{n-1}\end{pmatrix}$$

的秩为 n.

证明：

$$A\xrightarrow{r_2\leftrightarrow r_1}\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1\\ 0 & \cdots & \cdots & 0 & a_0\\ 0 & 1 & \cdots & 0 & a_2\\ \vdots & & & \vdots & \vdots\\ 0 & 0 & \cdots & 1\ 0 & a_{n-2}\\ 0 & 0 & \cdots & 0\ 1 & a_{n-1}\end{pmatrix}\xrightarrow{r_3\leftrightarrow r_2}\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1\\ 0 & 1 & \cdots & 0 & a_2\\ 0 & \cdots & \cdots & 0 & a_0\\ \vdots & & & \vdots & \\ 0 & 0 & \cdots & 0\ 1 & a_{n-1}\end{pmatrix}\xrightarrow{\cdots\cdots}$$

$$\xrightarrow{r_n\leftrightarrow r_{n-1}}\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1\\ 0 & 1 & \cdots & 0 & a_2\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0\ 1 & a_{n-1}\\ 0 & 0 & \cdots & 0\ 0 & a_0\end{pmatrix}$$

$a_0\neq0\Rightarrow \mathrm{rank}(A)=n$.

4. 补充：$\phi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射，ϕ 单 $\Rightarrow m\geq n$.

pf：ϕ 单 $\Rightarrow \ker(\phi)=\{\vec{0}\}\Rightarrow \dim(\mathrm{im}(\phi))=n$. 又 $\because \mathrm{im}(\phi)\subseteq\mathbb{R}^m$

$$\Rightarrow \underset{\overset{\|}{n}}{\dim(\mathrm{im}(\phi))}\leq m$$

$\phi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射，ϕ 满 $\Rightarrow n\geq m$

ϕ 满 $\Rightarrow \mathrm{im}(\phi)=\mathbb{R}^m\Rightarrow \dim(\mathrm{im}(\phi))=m\Rightarrow \dim(\ker(\phi))=n-m\geq0\Rightarrow n\geq m$.

证明：设 $\phi:\mathbb{R}^n\to\mathbb{R}^n$ 线性映射，且对 $\forall\vec{x}\in\mathbb{R}^n$，$\phi(\vec{x})\in\langle\vec{x}\rangle$ 证明：ϕ 是一个数乘映射，i.e. $\exists\lambda\in\mathbb{R}$, s.t. $\forall\vec{x}\in\mathbb{R}^n$, $\phi(\vec{x})=\lambda\vec{x}$.

Pf：设 $\vec{v_1},\cdots,\vec{v_n}$ 是 $\mathbb{R}^n$ 的一组基，且 $\phi(\vec{v_i})=\lambda_i\vec{v_i}$，$\lambda_i\in\mathbb{R}$，$i=1,\cdots,n$.

取 $\vec{x}=\vec{v_1}+\cdots+\vec{v_n}$，则 $\exists\lambda\in\mathbb{R}$, s.t. $\phi(\vec{x})=\lambda\vec{x}$

则 $\phi(\vec{v_1}+\cdots+\vec{v_n})=\phi(\vec{v_1})+\cdots+\phi(\vec{v_n})=\lambda_1\vec{v_1}+\cdots+\lambda_n\vec{v_n}=\lambda(\vec{v_1}+\cdots+\vec{v_n})$

$\Rightarrow(\lambda_1-\lambda)\vec{v_1}+\cdots+(\lambda_n-\lambda)\vec{v_n}=\vec{0}$

$\vec{v_1},\cdots,\vec{v_n}$ 线性无关 $\Rightarrow\lambda_1=\lambda_2=\cdots=\lambda_n=\lambda$.

$\forall y\in\mathbb{R}^n$，$\exists!\ a_1,\cdots,a_n\in\mathbb{R}$, s.t $\vec{y}=a_1\vec{v_1}+\cdots+a_n\vec{v_n}$

$\Rightarrow \phi(\vec{y})=\phi(a_1\vec{v_1}+\cdots+a_n\vec{v_n})=a_1\phi(\vec{v_1})+\cdots+a_n\phi(\vec{v_n})=\lambda(a_1\vec{v_1}+\cdots+a_n\vec{v_n})=\lambda\vec{y}$.

（线性映射基本定理）设$\vec{v_1},\cdots,\vec{v_n}$是$\mathbb{R}^n$的一组基，$\vec{w_1},\cdots,\vec{w_n}$是$\mathbb{R}^m$中的任意给定的向量，则$\exists!$的线性映射$\phi$，使得$\phi(\vec{v_j})=\vec{w_j}$，$j=1,2,\cdots,n$.

例 $\mathbb{R}^n$的任意子空间V都可以看成是某个齐次线性方程组的解空间.

Pf：claim：V是某个线性映射的核空间

设$\dim V=k\le n$，且$\vec{v_1},\cdots,\vec{v_k}$是$V$的一组基. 由基扩充定理，$\exists\ \vec{v}_{k+1},\cdots,\vec{v_n}\in\mathbb{R}^n$, s.t $\vec{v_1},\cdots,\vec{v_k},\vec{v}_{k+1},\cdots,\vec{v_n}$是$\mathbb{R}^n$的一组基. 由线性映射基本定理，可知，$\exists!$线性映射$\varphi:\mathbb{R}^n\to\mathbb{R}^?$ s.t

$$\varphi(\vec{v_i})=\begin{cases}\vec{0}, & 1\le i\le k\\ \vec{v_i}, & k+1\le i\le n\end{cases}$$

下面证明$V=\ker\varphi$.

由于$\vec{v_1},\cdots,\vec{v_k}\in\ker\varphi$，$\Rightarrow V\subset\ker(\varphi)$. $\Rightarrow \dim(\ker\varphi)\ge k$.

由$\vec{v}_{k+1},\cdots,\vec{v_n}\in\mathrm{im}\varphi$，$\Rightarrow \dim(\mathrm{im}\varphi)\ge n-k$.

由对偶定理可知，$\dim(\ker\varphi)+\dim(\mathrm{im}\varphi)=n$

$\Rightarrow \dim(\ker\varphi)\le k$.

$\Rightarrow \dim(\ker\varphi)=k$

$\Rightarrow V=\ker\varphi$

设A是φ在$\vec{e_1},\cdots,\vec{e_n}$下的矩阵，则$\forall\vec{x}\in\mathbb{R}^n$，$\varphi(\vec{x})=A\vec{x}$

$$\varphi(\vec{e_1},\cdots,\vec{e_n})=(\vec{e_1},\cdots,\vec{e_n})A$$

$\Rightarrow V=\ker\varphi=\{\vec{x}\in\mathbb{R}^n\mid A\vec{x}=\vec{0}\}$.

矩阵的运算

线性映射在标准基下的矩阵表示

$$\phi:\mathbb{R}^n\to\mathbb{R}^m$$

$\vec{e_1},\cdots,\vec{e_n}$是$\mathbb{R}^n$的标准基，

$\vec{\varepsilon_1},\cdots,\vec{\varepsilon_m}$是$\mathbb{R}^m$的标准基

设$\phi(\vec{e_j})=\sum_{i=1}^{m}a_{ij}\vec{\varepsilon_i}$

$$A_\phi:=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\a_{21}&\cdots&a_{2n}\\\vdots&&\vdots\\a_{m1}&\cdots&a_{mn}\end{pmatrix}$$ 称为ϕ在标准基下的矩阵（唯一确定）

Thm. 定义

$$\Phi: \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m\times n}$$
$$\phi \mapsto A_\phi,$$

$\{\vec{e_1}, \cdots, \vec{e_n}\}$ $\mathbb{R}^n$

$\{\vec{\varepsilon_1}, \cdots, \vec{\varepsilon_m}\}$ $\mathbb{R}^m$

$A_\phi = (\phi(\vec{e_1}), \cdots, \phi(\vec{e_n}))$.

则Φ是双射且逆为 $\Psi: \mathbb{R}^{m\times n} \mapsto \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$$A \mapsto \Phi_A$$

$A = (\vec{A}^{(1)}, \cdots, \vec{A}^{(n)})_{m\times n}$

$\psi(A)(\vec{e_j}) = \vec{A}^{(j)}$, $j = 1, 2, \cdots, n$

Thm. 设 $\varphi: \mathbb{R}^n \to \mathbb{R}^m$为线性映射，$A_{m\times n}$为$\varphi$在标准基下的矩阵，则

$$\dim(\ker\varphi) + \dim(\mathrm{im}\varphi) = n$$

$\dim(\mathrm{im}\varphi) = \dim(V_c(A)) = \mathrm{rank}(A)$

$\ker\phi := \{\vec{x} \mid A\vec{x} = \vec{0}\}$

注 φ 满射 $\Leftrightarrow$ $\mathrm{rank}(A) = m$，行满秩

φ 单射 $\Leftrightarrow$ $\mathrm{rank}(A) = n$，列满秩

φ 双射 $\Leftrightarrow$ $m = n$. A为满秩矩阵

例 $\phi: \mathbb{R}^5 \mapsto \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 + x_3 + x_4 + x_5 \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 \\ x_2 + 2x_3 + 2x_4 + 6x_5 \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 \end{pmatrix}$$

解：$\phi(\vec{e_1}) = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}, \vec{\varepsilon_4})\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$　　$\phi(\vec{e_4}) = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}, \vec{\varepsilon_4})\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

$\phi(\vec{e_2}) = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}, \vec{\varepsilon_4})\begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}$　　$\phi(\vec{e_5}) = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}, \vec{\varepsilon_4})\begin{pmatrix} 0 \\ -3 \\ 6 \\ -1 \end{pmatrix}$

$\phi(\vec{e_3}) = (\vec{\varepsilon_1}, \vec{\varepsilon_2}, \vec{\varepsilon_3}, \vec{\varepsilon_4})\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 & -3 \\ 0 & 1 & 2 & 2 & 6 \\ 5 & 4 & 3 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -2 & -6 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & -1 & -2 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank}(A) = 2 \Rightarrow \dim(\mathrm{im}(\phi)) = 2. \Rightarrow \dim(\ker(\phi)) = 5 - 2 = 3.$

$\ker(\phi)$对应的齐次线性方程组为

$$\begin{cases} x_1 = x_3 + x_4 + 5x_5 \\ x_2 = -2x_3 - 2x_4 - 6x_5. \end{cases}$$

∴ $\ker(\phi)$的一组基为 $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

此时 $\begin{vmatrix} \cos\alpha & -\sin\alpha \end{vmatrix}^{n}$

A中任意两个线性无关的列向量是im(φ)的一组基，如

$$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

注：初等行变换不改变列向量的线性关系。

矩阵的线性运算

$\phi, \psi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$，它们在标准基下的表示分别是 $A=(a_{ij})_{m\times n}$ 和 $B=(b_{ij})_{m\times n}$，则

$$(\phi+\psi)(e_j) = \phi(e_j) + \psi(e_j) = \vec{A}^{(j)} + \vec{B}^{(j)}, \ j=1,2,\dots,n.$$

定义

$$A+B := (\vec{A}^{(1)} + \vec{B}^{(1)}, \dots, \vec{A}^{(n)} + \vec{B}^{(n)}).$$

$$\lambda\phi(\vec{e_j}) = \lambda\phi(e_j) = \lambda\vec{A}^{(j)}, \ j=1,2,\dots n.$$

取.

定义

$$\lambda A = (\lambda\vec{A}^{(1)}, \dots, \lambda\vec{A}^{(n)})$$

矩阵乘法

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\psi} & \mathbb{R}^s \\ & \searrow^{\phi\circ\psi} & \downarrow \phi \end{array}$$

prop. $\underbrace{\psi}_{B} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$, $\underbrace{\phi}_{A} \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^m)$. 则 $\underbrace{\phi\circ\psi}_{C} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

$$\underbrace{i \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{is} \end{pmatrix}_{m\times s}}_{A} \underbrace{\begin{array}{c} j \\ \begin{pmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{js} \end{pmatrix}_{s\times n} \end{array}}_{B} = \underbrace{\begin{array}{c} j \\ \begin{pmatrix} \sum_{k=1}^{s} a_{ik} b_{kj} \end{pmatrix} i \end{array}}_{C}, \ \text{i.e.}\ c_{ij} = \sum_{k=1}^{s} a_{i,k} b_{k,j}, \ i=1,$$

prop. ① 结合律 $(A\cdot B)\cdot C = A\cdot(B\cdot C)$ $(A\in\mathbb{R}^{m\times s}, B\in\mathbb{R}^{s\times k}, C\in\mathbb{R}^{k\times n})$

② 分配律 $(A+B)\cdot C = A\cdot C + B\cdot C$ $(A, B\in\mathbb{R}^{m\times s}, C\in\mathbb{R}^{k\times n})$

$(A+B)\cdot C = A\cdot C + B\cdot C$

③ 数乘交换 $A\in\mathbb{R}^{m\times s}, B\in\mathbb{R}^{s\times n}$ $\alpha(A\cdot B) = A\cdot(\alpha B)$

④ 转置 $(A\cdot B)^t = B^t\cdot A^t$.

注意：交换律不一定成立，消去律也不一定成立 $\longrightarrow$ $A=\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

A·B有意义，B·A不一定有。

$C=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$AC=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $BC=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$AC=BC \not\Rightarrow B=A$

计算 $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n$.

$n=2$ 时, $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}=\begin{pmatrix}\cos^2\alpha-\sin^2\alpha & -2\sin\alpha\cos\alpha\\ 2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha\end{pmatrix}=\begin{pmatrix}\cos2\alpha & -\sin2\alpha\\ \sin2\alpha & \cos2\alpha\end{pmatrix}$

......

猜测 $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n=\begin{pmatrix}\cos n\alpha & -\sin n\alpha\\ \sin n\alpha & \cos n\alpha\end{pmatrix}$

证: $n=2$ ✓

假设 $n-1$ 成立, 则 $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^{n-1}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$

$$\underset{\text{归纳假设}}{=}\begin{pmatrix}\cos(n-1)\alpha & -\sin(n-1)\alpha\\ \sin(n-1)\alpha & \cos(n-1)\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

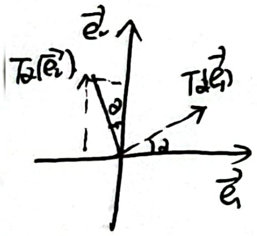
$$=\begin{pmatrix}\cos((n-1)\alpha+\alpha) & -\sin(\alpha+(n-1)\alpha)\\ \sin((n-1)\alpha+\alpha) & \cos((n-1)\alpha+\alpha)\end{pmatrix}$$

$$=\begin{pmatrix}\cos n\alpha & -\sin n\alpha\\ \sin n\alpha & \cos n\alpha\end{pmatrix}.$$

注: 设 $T_\alpha:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ 是旋转, 即

$A=(T_\alpha(\vec{e_1}),T_\alpha(\vec{e_2}))=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$, i.e. $T_\alpha(\vec{e_1})=\cos\alpha\,\vec{e_1}+\sin\alpha\,\vec{e_2}$

$T_\alpha(\vec{e_2})=-\sin\alpha\,\vec{e_1}+\cos\alpha\,\vec{e_2}$.



即 $\vec{e_1},\vec{e_2}$ 逆时针旋转 α.

$T_\alpha^k:\mathbb{R}^2\longrightarrow\mathbb{R}^2$. $k\geq1$.

$\vec{x}\longmapsto A^k\vec{x}$.

$\vec{e_1},\vec{e_2}$ 逆时针旋转 $k\alpha$.