1.
$$\phi: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1+x_2 \\ x_1-x_3 \\ x_2+x_3 \\ x_1+2x_2+x_3 \end{pmatrix}$$

Pf: (1) $\forall \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \in \mathbb{R}^3, \quad \alpha, \beta \in \mathbb{R}$, s.t.

$$\phi\left(\alpha\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}\right) = \phi\begin{pmatrix} \alpha x_1+\beta\tilde{x}_1 \\ \alpha x_2+\beta\tilde{x}_2 \\ \alpha x_3+\beta\tilde{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1+\beta\tilde{x}_1+\alpha x_2+\beta\tilde{x}_2 \\ \alpha x_1+\beta\tilde{x}_1-\alpha x_3-\beta\tilde{x}_3 \\ \alpha x_2+\beta\tilde{x}_2+\alpha x_3+\beta\tilde{x}_3 \\ \alpha x_1+\beta\tilde{x}_1+2\alpha x_2+2\beta\tilde{x}_2+\alpha x_3+\beta\tilde{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_1+x_2)+\beta(\tilde{x}_1+\tilde{x}_2) \\ \alpha(x_1-x_3)+\beta(\tilde{x}_1-\tilde{x}_3) \\ \alpha(x_2+x_3)+\beta(\tilde{x}_2+\tilde{x}_3) \\ \alpha(x_1+2x_2+x_3)+\beta(\tilde{x}_1+2\tilde{x}_2+\tilde{x}_3) \end{pmatrix}$$

$$= \alpha\phi\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) + \beta\phi\left(\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}\right)$$

$\Rightarrow \phi$是线性映射

(2) 设 ϕ在标准基下矩阵为A，

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

(3) 通过初等行变换得

$$A \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\because \mathrm{rank}(A) = 2$, $\therefore \dim(\mathrm{im}(\varphi)) = 2$，故 $\dim(\ker(\varphi)) = 3-2 = 1$

(4) $\ker(\varphi)$对应的齐次线性方程组为

$$\begin{cases} x_1+x_2=0 \\ x_2+x_3=0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow \ker(\varphi) = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

A中任意两个线性无关的列向量是 $\mathrm{im}(\varphi)$的一组基

$$\Rightarrow \mathrm{im}(\varphi) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle.$$

2.

(1) $$\begin{pmatrix} -1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & -3 \\ 2 & 5 \end{pmatrix}.$$

(2). $$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta-\sin\alpha\sin\beta & -\cos\alpha\sin\beta-\sin\alpha\cos\beta \\ \sin\alpha\cos\beta+\cos\alpha\sin\beta & -\sin\alpha\sin\beta+\cos\alpha\cos\beta \end{pmatrix}.$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta). & \cos(\alpha+\beta) \end{pmatrix}$$

3.(1) $\dim(V_A)=3$

$\operatorname{rank}(A) = 7-3 = 4$

(2) $0 \le \operatorname{rank}(A) \le 5$,

$7-5 \le \dim(V_A) = 7-\operatorname{rank}(A) \le 7-0$，即 $2 \le \dim(V_A) \le 7 \Rightarrow \dim(V_A) \ne 1$

4. 解：(1) $\forall \vec{y_1}, \vec{y_2} \in \mathbb{R}^m$，由于$\phi$是线性双射，故$\exists!\ \vec{x_1}, \vec{x_2} \in \mathbb{R}^n$，s.t $\phi(\vec{x_i}) = \vec{y_i}$，$i=1,2$. i.e. $\vec{x_i} = \phi^{-1}(\vec{y_i})$

$\forall \alpha, \beta \in \mathbb{R}$，$\phi(\alpha\vec{x_1}+\beta\vec{x_2}) = \alpha\phi(\vec{x_1})+\beta\phi(\vec{x_2}) = \alpha y_1 + \beta y_2$.

$\phi^{-1}(\alpha y_1+\beta y_2) = \phi^{-1}\phi(\alpha\vec{x_1}+\beta\vec{x_2}) = \alpha\vec{x_1}+\beta\vec{x_2} = \alpha\phi^{-1}(\vec{y_1})+\beta\phi^{-1}(\vec{y_2})$

$\Rightarrow \phi^{-1}$是线性映射.

(2) ϕ是线性单射，则 $\ker(\phi)=\{\vec{0}\}$，故 $\dim(\ker(\phi))=0$. 由对偶定理

$\dim(\ker(\phi)) + \dim(\operatorname{im}(\phi)) = n$

$\Rightarrow \dim(\operatorname{im}(\phi)) = n$.

由ϕ是线性满射，故 $\operatorname{im}(\phi)=\mathbb{R}^m$，从而 $\dim(\operatorname{im}(\phi))=m$.

$\Rightarrow m=n$.

5. (1)

$$AB = (\alpha_1,\cdots,\alpha_n)\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} = \left(\sum_{i=1}^n \alpha_i\beta_i\right)$$

$$BA = \begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}(\alpha_1,\cdots,\alpha_n) = \begin{pmatrix}\alpha_1\beta_1 & \cdots & \alpha_n\beta_1\\ \alpha_1\beta_2 & \cdots & \alpha_n\beta_2\\ \vdots & & \\ \alpha_1\beta_n & \cdots & \alpha_n\beta_n\end{pmatrix}$$

$$\operatorname{rank}(A) = \begin{cases} 0, & \alpha_1=\alpha_2=\cdots=\alpha_n=0.\\ 1, & \alpha_i\text{不全为}0.\end{cases}$$

$$\operatorname{rank}(B) = \begin{cases} 0, & \beta_1=\beta_2=\cdots=\beta_n=0\\ 1, & \beta_i\text{不全为}0.\end{cases}$$

(2) 证明：

设 $A=\begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{A_r}\\ \vec{A_{r+1}}\\ \vdots\\ \vec{A_m}\end{pmatrix}$，不妨设 $\vec{A_1},\cdots,\vec{A_r}$是$V_r(A)$的一组基，且

$\vec{A_i} = \alpha_{i,1}\vec{A_1}+\cdots+\alpha_{i,r}\vec{A_r}$，$i=r+1,\cdots,m$，$\alpha_{i,j}\in\mathbb{R}$.

$$\Rightarrow A = \begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{A_r}\\ \alpha_{r+1,1}\vec{A_1}+\cdots+\alpha_{r+1,r}\vec{A_r}\\ \vdots\\ \alpha_{m,1}\vec{A_1}+\cdots+\alpha_{m,r}\vec{A_r}\end{pmatrix} = \underbrace{\begin{pmatrix}\vec{A_1}\\ 0\\ \vdots\\ \vec{0}\\ \alpha_{r+1,1}\vec{A_1}\\ \vdots\\ \alpha_{m,1}\vec{A_1}\end{pmatrix}}_{B_1} + \underbrace{\begin{pmatrix}\vec{0}\\ \vec{A_2}\\ \vdots\\ 0\\ \alpha_{r+1,2}\vec{A_2}\\ \vdots\\ \alpha_{m,2}\vec{A_2}\end{pmatrix}}_{B_2} + \cdots + \underbrace{\begin{pmatrix}\vec{0}\\ \vec{0}\\ \vdots\\ \vec{A_r}\\ \alpha_{r+1,r}\vec{A_r}\\ \vdots\\ \alpha_{m,r}\vec{A_r}\end{pmatrix}}_{B_r}$$

i.e. $A = B_1 + B_2 + \cdots + B_r$ 且 $\mathrm{rank}(B_i) = 1$, $i = 1, 2, \cdots r$.

若 $A = B_1 + \cdots + B_{r-1}$, $\mathrm{rank}(B_i) = 1$, 则 $\mathrm{rank}(A) \le \mathrm{rank}(B_1) + \cdots + \mathrm{rank}(B_{r-1}) = r-1$ $\rightarrow\leftarrow$.

计算 $\mathrm{rank}(BA)$

注: 若 $A = 0$ 或 $B = 0$, 则 $BA = 0 \Rightarrow \mathrm{rank}(BA) = 0$

若 $A \ne 0$ 且 $B \ne 0$, 则 $\mathrm{rank}(A) = \mathrm{rank}(B) = 1$

$\mathrm{rank}(A) + \mathrm{rank}(B) - 1 \le \mathrm{rank}(BA) \le \min(\mathrm{rank}(A), \mathrm{rank}(B))$

$\Rightarrow 1 \le \mathrm{rank}(BA) \le 1$

$\Rightarrow \mathrm{rank}(BA) = 1$

期中复习

1. 线性方程组

$$L: \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{mn}x_n = b_m \end{cases}$$

系数矩阵

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

增广矩阵

$B = (A | \vec{b})$.

$L: A\vec{x} = \vec{b}$ $\qquad$ L 相容 $\Leftrightarrow \mathrm{rank}(A) = \mathrm{rank}(B)$

$H: A\vec{x} = \vec{0}$ $\qquad$ L 确定 $\Leftrightarrow \mathrm{rank}(A) = \mathrm{rank}(B) = n$

对偶原理: $\dim(\mathrm{sol}(H)) + \mathrm{rank}(A) = n$.

若 L 相容, 则 $\mathrm{sol}(L) = \vec{v} + \mathrm{sol}(H)$, $\vec{v}$ 是 L 的一个特解.

2. 集合与映射

$f: S \to T$, $\forall x \in S$, $\exists!\, y \in T$, s.t. $f(x) = y$

$\mathrm{im} f := f(S)$

$f(S') = \{f(x) | x \in S'\}$, $\forall S' \subset S$

$f^{-1}(T') = \{x \in S | f(x) \in T'\}$,

单射: $\forall x_1, x_2 \in S$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

满射: $\forall t \in T$, $\exists x \in S$, s.t. $f(x) = t$

3. 等价关系 $\quad$ S 非空集合, $\sim$ 是 S 上的二元关系, 如果

① 自反性: $\forall x \in S$, $x \sim x$

② 对称性: $x, y \in S$ 且 $x \sim y$, 则 $y \sim x$

③ 传递性: 设 $x, y, z \in S$, $x \sim y$ 且 $y \sim z$, 则 $x \sim z$. 则称 $\sim$ 是等价关系

4.带余除法：设 $x\in\mathbb{Z}$, $m\in\mathbb{Z}^+$, 则 $\exists!\, q, r\in\{0,1,\cdots,m-1\}$, s.t $x=qm+r$.
$r=\mathrm{rem}(x,m)$, $q=\mathrm{quo}(x,m)$.

5.置换

$T=\{1,2,\cdots,n\}$, $\sigma: T\to T$ 双射，称为一个置换，记为 $\sigma=\begin{pmatrix}1 & 2 & \cdots & n\\ i_1 & i_2 & \cdots & i_n\end{pmatrix}$

循环分解：$\sigma=\tau_1\tau_2\cdots\tau_s$，不相交循环之积 设 l_i 为 τ_i 的长度.

$\mathrm{ord}(\sigma)=\sigma$ 的阶（i.e 使得 $\sigma^k=e$ 的最小正整数 k）
$= \mathrm{lcm}(l_1,l_2,\cdots,l_s)$.

ε_σ：σ 的符号(i.e.，可分解奇/偶个对换之积)，$\varepsilon_\sigma=(-1)^{\sum_{i=1}^{s}(l_i-1)}$

6.整数的算术.

若 $d\mid m$ 且 $d\mid n$，则 $d\mid um+vn$.

gcd, lcm, 设 $m,n\in\mathbb{Z}^+$, $\boxed{\exists\, u,v\in\mathbb{Z}, \text{ s.t } \gcd(m,n)=um+vn.}$

Bezout 等式

$\mathrm{lcm}(m,n)=\dfrac{|mn|}{\gcd(m,n)}$ 如何计算 $\gcd(m,n)$, m,n，利用扩展 Euclidean 算法

prop, $m,n\in\mathbb{Z}$, $\gcd(m,n)=1 \Leftrightarrow \exists\, u,v\in\mathbb{Z}$, s.t $um+vn=1$.

7.向量空间

① 线性组合：$\exists\, \alpha_1,\cdots,\alpha_k\in\mathbb{R}$, s.t $\vec{w}=\alpha_1\vec{v_1}+\cdots+\alpha_k\vec{v_k}$

线性相关：$\exists$ 不全为 0 的 $\alpha_1,\cdots,\alpha_k$, s.t $\alpha_1\vec{v_1}+\cdots+\alpha_k\vec{v_k}=\vec{0}$.

线性无关：如果 $\exists\, \alpha_1,\cdots,\alpha_k\in\mathbb{R}$, s.t $\alpha_1\vec{v_1}+\cdots+\alpha_k\vec{v_k}=\vec{0}$，则 $\alpha_1=\cdots=\alpha_k=0$.

L: $A\vec{x}=\vec{b}$, $A=(\vec{A^{(1)}},\cdots,\vec{A^{(n)}})$

H: $A\vec{x}=\vec{0}$

L 相容 $\Leftrightarrow$ $\vec{b}$ 是 $\vec{A^{(1)}},\cdots,\vec{A^{(n)}}$ 的线性组合

H 有非零解 $\Leftrightarrow$ $\vec{A^{(1)}},\cdots,\vec{A^{(n)}}$ 线性相关.

prop ① $\vec{v_1},\cdots,\vec{v_i}$ 线性相关 $\Rightarrow$ $\vec{v_1},\cdots,\vec{v_i},\cdots,\vec{v_k}$ 线性相关.

② $\vec{v_1},\cdots,\vec{v_i},\cdots,\vec{v_k}$ 线性无关 $\Rightarrow$ $\vec{v_1},\cdots,\vec{v_i}$ 线性无关

③ $\forall$ $\begin{pmatrix}x_{1,1}\\ \vdots\\ x_{1,m}\end{pmatrix}, \begin{pmatrix}x_{2,1}\\ \vdots\\ x_{2,m}\end{pmatrix},\cdots,\begin{pmatrix}x_{n,1}\\ \vdots\\ x_{n,m}\end{pmatrix}\in\mathbb{R}^m$ 线性无关，则 将其扩展后成为

$$\begin{pmatrix}x_{1,1}\\ \vdots\\ x_{1,m}\\ x_{1,m+1}\\ \vdots\\ x_{1,s}\end{pmatrix}, \begin{pmatrix}x_{2,1}\\ \vdots\\ x_{2,m}\\ x_{2,m+1}\\ \vdots\\ x_{2,s}\end{pmatrix},\cdots,\begin{pmatrix}x_{n,1}\\ \vdots\\ x_{n,m}\\ x_{n,m+1}\\ \vdots\\ x_{n,s}\end{pmatrix}\in\mathbb{R}^s \text{ 线性无关.}$$

① $\vec{v_1},\cdots,\vec{v_k}$ 线性相关 $\Leftrightarrow \exists \vec{v_i}, i\in\{1,\cdots k\}$, s.t $\vec{v_i}$ 是 $\vec{v_1},\cdots,\vec{v_{i-1}},\vec{v_{i+1}},\cdots,\vec{v_k}$ 的线性组合.

(线性组合引理)

$\vec{w_1},\cdots,\vec{w_l}$ 是 $\vec{v_1},\cdots,\vec{v_k}$ 线性组合, $l>k$, 则 $\vec{w_1},\cdots,\vec{w_l}$ 线性相关.

eg $\mathbb{R}^n$ 中任意 $n+1$ 个向量线性相关.

7 子空间:

U 是 $\mathbb{R}^n$ 中的非空子集, $\forall \vec{x},\vec{y}\in U$, $\alpha\in\mathbb{R}$, s.t $\vec{x}+\vec{y}\in U$, $\alpha\vec{x}\in U$.

eg. 设 $p,q\in\mathbb{Q}$, 不全为 0, $S=\left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \middle| p\alpha+q\beta=0, \alpha,\beta\in\mathbb{Q}\right\}$

S 是否是 $\mathbb{R}^n$ 的子空间?

解: 不是, $\begin{pmatrix}-q\\ p\end{pmatrix}\in S$, $\sqrt{2}\begin{pmatrix}-q\\ p\end{pmatrix}\notin S$.

$U_1,U_2\subset\mathbb{R}^n$ 子空间, 则 $U_1\cap U_2$, U_1+U_2 也是子空间.

$\langle\vec{v_1},\cdots,\vec{v_k}\rangle=\{\alpha_1\vec{v_1}+\cdots+\alpha_k\vec{v_k} \mid \alpha_i\in\mathbb{R}\}$ 是包含 $\vec{v_1},\cdots,\vec{v_k}$ 的最小子空间

8 基与维数

$\vec{u_1},\cdots,\vec{u_k}\in U$ 是 U 的一组基, ① $\vec{u_1},\cdots,\vec{u_k}$ 线性无关

② $\forall\vec{u}\in U$, $\vec{u}\in\langle\vec{u_1},\cdots,\vec{u_k}\rangle$,

$\Rightarrow U=\langle\vec{u_1},\cdots,\vec{u_k}\rangle$

基的个数称为维数, 记为 $\dim(U)$.

基扩充定理: $\vec{u_1},\cdots,\vec{u_k}$ 是子空间 U 线性无关的向量, 则 U 有一组基包含 $\vec{u_1},\cdots,\vec{u_k}$.

维数公式: $\dim(U+W)+\dim(U\cap W)=\dim(U)+\dim(W)$

直和: $U\cap W=\{0\}\Leftrightarrow U+W=U\oplus W$,

$\Leftrightarrow \dim(U+W)=\dim(U)+\dim(W)$.

8. 矩阵的秩

$A_{m\times n}=\begin{pmatrix}\vec{A_1}\\ \vdots\\ \vec{A_m}\end{pmatrix}=(\vec{A^{(1)}},\cdots,\vec{A^{(n)}})$.

$V_r(A)=\langle\vec{A_1},\cdots,\vec{A_m}\rangle$.

$V_c(A)=\langle\vec{A^{(1)}},\cdots,\vec{A^{(n)}}\rangle$.

$\operatorname{rank}(A)=\dim(V_r(A))=\dim(V_c(A))$

$A\in\mathbb{R}^{m\times n}$

$\operatorname{rank}(A)\le\min(m,n)$

$\operatorname{rank}(A+B)\le\operatorname{rank}(A)+\operatorname{rank}(B)$.

转置 A^t: $(A^t)^t=A$, $\operatorname{rank}(A^t)=\operatorname{rank}(A)$

$(AB)^t=B^tA^t$

9. 线性映射

$\varphi: \mathbb{R}^n \to \mathbb{R}^m$, $\varphi(\vec{x}+\vec{y}) = \varphi(\vec{x})+\varphi(\vec{y})$, $\varphi(\alpha\vec{x}) = \alpha\varphi(\vec{x})$. $\vec{x},\vec{y}\in\mathbb{R}^n$, $\alpha\in\mathbb{R}$.

prop. $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射

① $\vec{v_1},\dots,\vec{v_k}$ 线性相关, 则 $\varphi(\vec{v_1}),\dots,\varphi(\vec{v_k})$ 线性相关

② $U\subset\mathbb{R}^n$ 子空间, $\varphi(U)$ 是 $\mathbb{R}^m$ 子空间, 特别地, $\mathrm{im}(\varphi)$ 是 $\mathbb{R}^m$ 的子空间.

③ W 是 $\mathbb{R}^m$ 子空间, $\varphi^{-1}(W)$ 是 $\mathbb{R}^n$ 子空间, 特别地, $\underbrace{\varphi^{-1}(\vec{0})}_{\ker\varphi}$ 是 $\mathbb{R}^n$ 的子空间.

例. 设 $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射, 证明: 如果 φ 是满射, 则对 $\mathbb{R}^m$ 的任意子空间 W, 有

$$\dim(\varphi^{-1}(W)) \geq \dim(W)$$

prop. $\vec{v_1},\dots,\vec{v_n}$ 是 $\mathbb{R}^n$ 的一组基, $\vec{w_1},\dots,\vec{w_n}$ 是 $\mathbb{R}^m$ 中任意给定的向量, 则 $\exists!$ 线性映射 φ s.t. $\varphi(\vec{v_j})=\vec{w_j}$, $j=1,\dots,n$.

$(\mathrm{im}(\varphi) = \langle\varphi(\vec{v_1}),\dots,\varphi(\vec{v_n})\rangle = \langle\vec{w_1},\dots,\vec{w_n}\rangle$

对偶定理: $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射, $\dim(\ker(\varphi))+\dim(\mathrm{im}(\varphi))=n$.

线性映射与矩阵一一对应

$\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$: $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 线性映射集合

$$\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)\longleftrightarrow\mathbb{R}^{m\times n}$$

$\varphi \longmapsto A_\varphi$ $A_\varphi := (\varphi(\vec{e_1}),\dots,\varphi(\vec{e_n}))$

$\varphi_A \longleftarrow A$ $\varphi_A(\vec{e_j}) = \vec{A}^{(j)}$, $j=1,2,\dots n$.

$\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \longmapsto A\vec{x}$, $A\in\mathbb{R}^{m\times n}$.

prop $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ 线性映射, 在标准基下矩阵 $A\in\mathbb{R}^{m\times n}$, 对应齐次线性方程组 H

① $\mathrm{im}(\varphi)=V_C(A)$, $\dim(\mathrm{im}(\varphi))=\mathrm{rank}(A)$, 特别地, φ 满 $\Leftrightarrow$ A 行满秩

② $\ker(\varphi)=\mathrm{sol}(H)$, $\dim(\ker(\varphi))=n-\mathrm{rank}(A)$, 特别地 φ 单 $\Leftrightarrow$ A 列满秩

③ φ 双 $\Leftrightarrow$ $m=n$ 且 A 满秩.

eg 给定线性映射 φ, 计算 φ 在标准基下的矩阵, $\dim(\ker\varphi)$, $\dim(\mathrm{im}\,\varphi)$, $\ker(\varphi)$ 的一组基, $\mathrm{im}(\varphi)$ 的一组基.

画出线性映射的定义矩阵.

$\varphi \xrightarrow{对应} A$

$\psi \xrightarrow{对应} B$

$\varphi, \psi \in Hom(\mathbb{R}^n; \mathbb{R}^m)$, $A, B \in \mathbb{R}^{m\times n}$.

$\varphi + \psi \to A+B$.

$\lambda\varphi \longrightarrow \lambda A$, $\lambda \in \mathbb{R}$

$\varphi \in Hom(\mathbb{R}^s, \mathbb{R}^n)$, $\psi \in Hom(\mathbb{R}^n, \mathbb{R}^s)$

$A \in \mathbb{R}^{m\times s}$ $\quad B \in \mathbb{R}^{s\times n}$

$\varphi \circ \psi \longrightarrow AB$ $\quad i\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{is} \end{pmatrix}_{m\times s} \begin{pmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{js} \end{pmatrix}_{s\times n} = i\begin{pmatrix} \cdots & c_{ij} & \cdots \end{pmatrix}_{m\times n}$

$\quad A \quad\quad B \quad\quad AB$, $c_{ij} = \sum_{k=1}^{s} a_{ik} b_{ks}$

eg. 考虑齐次线性方程组 H

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

设 $V_i = sol(a_{i1}x_1 + \cdots + a_{in}x_n = 0)$, $i=1,2,\cdots,m$. 则 $sol(H) = \bigcap_{i=1}^{m} V_i$

Pf: 设 $\vec{v} = (v_1, \cdots, v_n) \in sol(H)$, 则对任意 $i \in \{1,\cdots,m\}$, 有

$$a_{i1}v_1 + \cdots + a_{in}v_n = 0$$

即 $\vec{v} \in V_i$, $i=1,\cdots,m$.

$$\Rightarrow \vec{v} \in \bigcap_{i=1}^{n} V_i$$

若 $\vec{v} = (v_1, \cdots, v_n) \in \bigcap_{i=1}^{m} V_i$, 则对任意 $i \in \{1,\cdots,m\}$, 有 $\vec{v} \in V_i$, 即

$$a_{i1}v_1 + \cdots + a_{in}v_n = 0$$

$\Rightarrow \vec{v} \in sol(H)$.

综上, $\bigcap_{i=1}^{m} V_i = sol(H)$