行列式乘积定理

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回忆：设 $A \in M_n(\mathbb{R})$

$$\det(A) = \sum_{k=1}^{n} a_{ik} A_{ik}$$

$$= \sum_{k=1}^{n} a_{kj} A_{kj}$$

$\rightarrow$ $A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$, $\det(A) = a_{11} a_{22} \cdots a_{nn}$

§3.2 分块矩阵的行列式

定理：设 $A \in M_m(\mathbb{R})$，$B \in M_n(\mathbb{R})$

$C \in \mathbb{R}^{m \times n}$.

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}_{(m+n) \times (m+n)}$$

则 $\det(M) = \det(A) \det(B)$

证：（数学归纳）对 m 归纳

$m=1$

$$M = \begin{pmatrix} \boxed{a} & c_1 \cdots & c_n \\ 0 & & \\ \vdots & & B \\ 0 & & \end{pmatrix}$$

$\det(M) = a \det(B) = \det(A) \det(B)$.

$\det(M) = a\det(B) = \det(A)\det(B)$.

设 $m>1$ 且 $m-1$ 时结论成立

考虑 m.

$$\det(M) = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}} & \begin{matrix} a_{12} \cdots a_{1m} \\ a_{22} \cdots a_{2m} \\ \vdots \\ a_{m2} \cdots a_{mm} \end{matrix} & C \\ 0 & & B \end{pmatrix}$$

$= a_{11} A_{11} \det(B) + a_{21} A_{21} \det B + \cdots + a_{m1} A_{m1} \det(B)$

A_{11} 关于 A 的

$= (a_{11} A_{11} + a_{21} A_{21} + \cdots + a_{m1} A_{m1}) \det(B)$

$= \det(A)\det(B)$.

(映射角度)

$f: \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{m} \longrightarrow \mathbb{R}$

$(\vec{A}^{(1)}, \cdots, \vec{A}^{(m)}) \longmapsto \det(M) = \det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$

可直接验证 f 是 m 重线性的 斜对称

由此可知

$\boxed{f = \omega \det(A)}$,

$$\boxed{f = w\,\det(A)},$$

根据行列式公式在第一节的推导过程

$w = f(\vec{e}_1, \cdots, \vec{e}_m)$，其中

$\vec{e}_1, \cdots, \vec{e}_m$ 是 $\mathbb{R}^m$ 的标准基

$$w = f(\vec{e}_1, \cdots, \vec{e}_m) = \det\begin{pmatrix} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & C \\ 0 & B \end{pmatrix}$$

按第1列，第2列，…第m列展开得

$w = \det(B)$

故 $\underbrace{f(\vec{A}^{(1)}, \cdots, \vec{A}^{(m)})}_{\det(M)} = \det(B)\det(A)$

由此可知， $\det(M) = \det(A)\det(B)$

$\square$

例 计算

$$D = \begin{vmatrix} 0 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{vmatrix}$$

$$= -\begin{vmatrix} 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix} = -36$$

$$= -\begin{vmatrix} 9 & 10 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix} = -36$$

推论：设 $A \in M_m(\mathbb{R})$，$B \in M_n(\mathbb{R})$
$C \in \mathbb{R}^{m\times n}$

$$\det\begin{pmatrix} C & A \\ B & 0 \end{pmatrix} = \underline{(-1)}^{mn}\det(A)\det(B)$$

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§3.3 乘积定理

定理：设 $A, B \in M_n(\mathbb{R})$

则 $\det(AB) = \underline{\det(A)\det(B)}$

证：(矩阵方法) 如果 A 不满秩

则 $\det(A) = 0$

且 AB 也是不满秩的

故 $\det(AB) = 0 = \det(A)\det(B)$

只要考虑 A 满秩

于是 A 是若干初等矩阵

之积，

引理：设 $C, M \in M_n(\mathbb{R})$

且 C 是初等阵，则

$$\det(CM) = \det(C)\det(M) = \det(MC)$$

引理的证明

情形1 $C = F_{ij}$

$\det(CM) = -\det(M)$

$\det(C) = -1$ $\quad \det(C)\det(M) = -\det(M)$

情形2 $C = F_{ij}(k)$，k 任意

$\det(CM) = \det(M)$

$\det(C) = 1 \Rightarrow \det(C)\det(M) = \det(M)$

情形3 $C = F_i(\lambda)$，$\lambda \neq 0$

$\det(CM) = \lambda \det(M)$

$\det(C) = \lambda \Rightarrow \det(C)\det(M) = \lambda\det(M)$

MC 的情形类似，引理成立

设 $A = \underbrace{C_1 \cdots C_p}$ $\quad C_1, \cdots, C_p$ 是

初等矩阵

$$\det(AB) = \det(C_1 \cdots C_p B)$$

$$= \det(C_1 (C_2 \cdots C_p) B)$$

$$= \det(C_1)\det(C_2 \cdots C_p B)$$

$$= \underbrace{\det(C_1)\det(C_2)}\det(C_3 \cdots C_p B)$$

$$= \det(C_1 C_2)\det(C_3 \cdots C_p B)$$

$$\vdots$$

$$= \det(C_1 C_2 \cdots C_p)\det(B)$$

$$= \det(A)\det(B) \qquad \square$$

映射法

$$f:\ \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n} \longrightarrow \mathbb{R}$$

$$\vec{B}^{(1)}, \cdots, \vec{B}^{(n)} \longmapsto \det(AB)$$

$$\|$$

$$\det(A\vec{B}^{(1)}, \cdots, A\vec{B}^{(n)})$$

$$f(\vec{B}^{(2)}, \vec{B}^{(1)}, \vec{B}^{(3)}, \cdots, \vec{B}^{(n)})$$

$$= \det(A\vec{B}^{(2)}, A\vec{B}^{(1)}, A\vec{B}^{(3)}, \cdots, A\vec{B}^{(n)})$$

$$= -f(\vec{B}^{(1)}, \vec{B}^{(2)}, \cdots, \vec{B}^{(n)})$$

$$A(\alpha\vec{u}+\beta\vec{v}) = \alpha A\vec{u}+\beta B\vec{v}$$

由上述说明可知，f 是 n 重线性

反对称的

f 是：由第一节行列式公式推导过程可知

$$f(\vec{B}^{(1)},\cdots,\vec{B}^{(n)}) = w\det(B)$$

又 $w = f(\vec{e}_1,\cdots,\vec{e}_n)$，其中 $\vec{e}_1\cdots\vec{e}_n$ 是 $\mathbb{R}^n$ 的标准基

$$w = \det(A(\vec{e}_1,\cdots,\vec{e}_n)) = \det(AE) = \det(A)$$

故 $f(\vec{B}^{(1)},\cdots,\vec{B}^{(n)}) = \det(A)\det(B)$

$$\|$$

$$\det(AB) \qquad \blacksquare$$

注：行列式的交换不变量

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

$| \ 1 \quad \cos\theta_1, \quad \cos 2\theta_1 \ |$

例: $D = \begin{vmatrix} 1 & \cos\theta_1 & \cos 2\theta_1 \\ 1 & \cos\theta_2 & \cos 2\theta_2 \\ 1 & \cos\theta_3 & \cos 2\theta_3 \end{vmatrix}$

解 $D = \begin{vmatrix} 1 & \cos\theta_1 & 2\cos^2\theta_1 - 1 \\ 1 & \cos\theta_2 & 2\cos^2\theta_2 - 1 \\ 1 & \cos\theta_3 & 2\cos^2\theta_3 - 1 \end{vmatrix}$

$= \left| \begin{pmatrix} 1 & \cos\theta_1 & \cos^2\theta_1 \\ 1 & \cos\theta_2 & \cos^2\theta_2 \\ 1 & \cos\theta_3 & \cos^2\theta_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right|$

$= \underline{\begin{vmatrix} 1 & \cos\theta_1 & \cos^2\theta_1 \\ 1 & \cos\theta_2 & \cos^2\theta_2 \\ 1 & \cos\theta_3 & \cos^2\theta_3 \end{vmatrix}} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$ [范德蒙德]

$= 2(\cos\theta_2 - \cos\theta_1)(\cos\theta_3 - \cos\theta_1)(\cos\theta_3 - \cos\theta_2)$ ✓

例: $A = \left(\underline{(\alpha_i + \beta_j)}^{n-1}\right)_{n\times n}$

$i, j \in \{1, 2, \ldots n\}$

解 $(\alpha_i + \beta_j)^{n-1}$

$= \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha_i^k \beta_j^{n-1-k}$

$= \underline{\left(\binom{n-1}{0}\alpha_i^0, \binom{n-1}{1}\alpha_i^1, \ldots \binom{n-1}{n-1}\alpha_i^{n-1} \right)} \begin{pmatrix} \beta_j^{n-1} \\ \beta_j^{n-2} \\ \vdots \\ \beta_j^0 \end{pmatrix}$

令

$$B = \begin{pmatrix} \binom{n-1}{0}\alpha_1^0 & \binom{n-1}{1}\alpha_1^1 & \cdots & \binom{n-1}{n-1}\alpha_1^{n-1} \\ \binom{n-1}{0}\alpha_2^0 & \binom{n-1}{1}\alpha_2^1 & \cdots & \binom{n-1}{n-1}\alpha_2^{n-1} \\ \vdots & \vdots & & \vdots \\ \binom{n-1}{0}\alpha_n^0 & \binom{n-1}{1}\alpha_n^1 & \cdots & \binom{n-1}{n-1}\alpha_n^{n-1} \end{pmatrix}$$

$$C = \begin{pmatrix} \beta_1^{n-1} & \beta_2^{n-1} & \cdots & \beta_n^{n-1} \\ \beta_1^{n-2} & \beta_2^{n-2} & & \beta_n^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & & 1 \end{pmatrix}$$

则 $A = BC$

$\det(A) = \det(B)\det(C)$

$\det(B) = \binom{n-1}{0}\binom{n-1}{1}\cdots\binom{n-1}{n-1}\boxed{V_n(\alpha_1\cdots\alpha_n)}$

$\det(C) = (-1)^{\frac{n(n-1)}{2}}\boxed{V_n(\beta_1,\cdots\beta_n)}$

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§3.4 伴随矩阵

设 $i,j \in \{1,2,\ldots,n\}$ Kronecker符号

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

利用此符号. $E_n = (\delta_{ij})_{n\times n}$

$(a_{11}\ \cdots\ a_{1n})$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A^t=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{pmatrix}$$

$$A^\vee :=\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$ 称为

A 的伴随矩阵 (adjoint matrix)

(通常记为 A^*)

定理：设 $A \in M_n(\mathbb{R})$

则 $\underline{A^\vee}\,\underline{A} = AA^\vee = \begin{pmatrix} |A| & & 0 \\ & \ddots & \\ 0 & & |A| \end{pmatrix}$

$= |A| E_n$

证：断言：$\forall\, i,j \in \{1,2,\cdots,n\}$

$\sum_{k=1}^{n} a_{i,k} A_{j,k} = \delta_{i,j}|A|$，$\sum_{k=1}^{n} a_{k,j} A_{k,i} = \delta_{i,j}|A|$

断言的证明：

情形1，$i=j$

j ..

情形1，$i=j$

$$\sum_{k=1}^{n} a_{i,k}A_{j,k} = \sum_{k=1}^{k} a_{i,k}A_{i,k} = |A|$$
$$= \delta_{i,i}|A|$$

情形2，$i\neq j$

$$\sum_{k=1}^{n} a_{i,k}A_{jk} = \begin{array}{c|c} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \underbrace{a_{i1}} & \underbrace{a_{i2}} & \cdots & \underbrace{a_{in}} \\ & \vdots & & \\ \underbrace{a_{i1}} & \underbrace{a_{i2}} & \cdots & \underbrace{a_{in}} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{n,n} \end{vmatrix} & \begin{array}{c} \\ \\ -i \\ \\ -j \\ \\ \\ \end{array} \end{array}$$

（按第j行展开）

$$= 0 = \delta_{i,j}|A|.$$ $\square$

另一个等式类似

$AA^{\vee}$中第i行第j列元素

$$(a_{i1}, a_{i2}, \cdots a_{in})\begin{pmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{pmatrix}$$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{i,n}A_{jn}$$
$$= \delta_{ij}|A| \quad [\text{引理}]$$

$$AA^{\vee} = (\delta_{i,j}|A|)_{n\times n} = \begin{pmatrix} |A| & & 0 \\ & \ddots & \\ 0 & & |A| \end{pmatrix}$$ $\square$

定理：设 $A \in M_n(R)$ 可逆，则

$$A^{-1}\frac{1}{|A|} \quad A^{\vee} \qquad A^{-1} = \frac{A^{\vee}}{|A|}$$

$\gamma^{\vee}\ |A|$

证: $A^{\vee}A = |A|E$

$\because |A| \neq 0 \quad \therefore \underbrace{\left(\frac{1}{|A|}A^{\vee}\right)}\underline{A} = E$

$$\Rightarrow A^{-1} = \frac{1}{|A|}A^{\vee} \quad \boxed{}$$

注: 上述定理得

$$\rightarrow A^{-1} = \begin{pmatrix} \frac{A_{11}}{|A|} & \cdots & \frac{A_{n1}}{|A|} \\ & \vdots & \\ \frac{A_{1n}}{|A|} & \cdots & \frac{A_{nn}}{|A|} \end{pmatrix}$$

设 $a_{ij} \in \mathbb{Z}$, $i, j \in \{1, 2 \cdots n\}$

A^{-1} 中有理数的公分~~母~~是 $|A|$

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§4 行列式的应用

§4.1 Cramer 法则

定理: 设 $A = (a_{ij}) \in M_n(\mathbb{R})$.

$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$. $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

则 $A\vec{x} = \vec{b}$ 有唯一解

$\Leftrightarrow A$ 可逆 ?

则 $A\vec{x}=\vec{0}$ 只有 ...

$\Longleftrightarrow A$ 可逆

第 i 列换为 $\vec{b}$

此时

$$x_i = \frac{\det(\vec{A}^{(1)},\dots,\vec{A}^{(i-1)},\vec{b},\vec{A}^{(i+1)},\dots,\vec{A}_n)}{\det(A)}$$

$i=1,\dots,n$

证: 设 A 可逆

则 $A\vec{x}=\vec{b} \Rightarrow \vec{x}=A^{-1}\vec{b}$

$$\Rightarrow \vec{x} = \frac{1}{|A|}A^{\vee}\vec{b}$$

$$= \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ & \vdots & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_i = \frac{1}{|A|}(b_1A_{1i}+\cdots+b_nA_{ni})$$

$$= \frac{1}{|A|}\det(\vec{A}^{(1)},\dots,\vec{A}^{(i-1)},\vec{b},\vec{A}^{(i+1)},\dots,\vec{A}^{(n)})$$

□

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§4.2 子式和矩阵的秩

§4.2 子式和矩阵的秩

定义：设 $A=(a_{ij})\in\mathbb{R}^{m\times n}$

$i_1,\dots,i_k\in\{1,\dots,m\}$
$j_1,\dots,j_k\in\{1,\dots,n\}$ } 不必依序两两不同

则

$$\det\begin{pmatrix} a_{i_1,j_1}, & a_{i_1,j_2}, & \cdots & a_{i_1,j_k} \\ a_{i_2,j_1}, & a_{i_2,j_2} & \cdots & a_{i_2,j_k} \\ & \cdots & & \\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_k} \end{pmatrix}$$

称为 A 的一个 k 阶子式 (minor)

记为 $M_A\begin{pmatrix} i_1, & i_2 & \cdots & i_k \\ j_1, & j_2 & \cdots & j_k \end{pmatrix}$

或 $M\begin{pmatrix} i_1, & i_2, & \cdots & i_k \\ j_1, & j_2 & \cdots & j_k \end{pmatrix}$

例：$A=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}_{3\times 4}$

$$M_A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}=\det\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

$$M_A\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}=\det\begin{pmatrix} 7 & 8 \\ 3 & 4 \end{pmatrix}$$

定理：设 $A \in \mathbb{R}^{m\times n}$ 非零，则下列命题等价

(i) $\operatorname{rank}(A) = r$

(ii) A 中所有大于 r 阶的子式都等于零，且存在一个 r 阶子式非零

(iii) A 中所有 $r+1$ 阶子式为零，且存在一个 r 阶子式非零

书上习题　设 $A, B \in M_n(\mathbb{R})$

证明　$(AB)^{\vee} = B^{\vee}A^{\vee}$

A 可逆时　容易

A 不可逆？

扰动