1、上周作业

$$\mathrm{rank}\begin{pmatrix} 8 & 2 & 2 & -1 \\ 1 & 7 & 4 & -2 \\ -2 & 4 & 2 & -1 \end{pmatrix} = 2 \qquad \mathrm{rank}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 3$$

故 $\dim\langle v_1, v_2, v_3, v_4\rangle = 2$，$u_1, u_2, u_3$ 为一组基

2、易证 $\varphi_i \in (\mathbb{R}[x]^{(n)})^*$

$i<j$ 时 $\varphi_i(x^j) = \frac{j!}{(j-i)!\,i!} x^j \Big|_{x=0} = 0$

$i=j$ 时 $\varphi_i(x^j) = 1\Big|_{x=0} = 0$

$i>j$ 时 $\varphi_i(x^j) = 0\Big|_{x=0} = 0$

故为对偶基

3、矩阵为 $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 5 & 0 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}$ $\mathrm{rank}\,A = 4$

4、$m - \mathrm{rank}(E_m - AB) = n - \mathrm{rank}(E_n - BA)$

将 A, B 看作线性映射，即证

$$\dim\ker(\mathrm{id}_{F^m} - AB) = \dim\ker(\mathrm{id}_{F^n} - BA)$$

注意到 $\forall \alpha \in \ker(\mathrm{id}_{F^m} - AB)$ 有

$$(\mathrm{id}_{F^n} - BA)B\alpha = B\alpha - B(AB\alpha) = B\alpha - B\alpha = 0$$

定义 φ: $\ker(\mathrm{id}_{F^m} - AB) \to \ker(\mathrm{id}_{F^n} - BA)$

$$\alpha \longmapsto B\alpha$$

由于 $\varphi = B|_{\ker(\mathbb{I}_{F^m}-AB)}$ 故 φ 为线性映射.

$\forall \alpha \in \ker(\mathbb{I}_{F^n}-BA)$, $\alpha = \varphi(A\alpha) \Rightarrow \varphi$ 为满射

$\forall \alpha \in \ker\varphi \Rightarrow B\alpha = 0 \Rightarrow \alpha = AB\alpha = 0$ 故 φ 为单射

综上, φ 为同构, $\dim\ker(\mathbb{I}_{F^m}-AB) = \dim\ker(\mathbb{I}_{F^n}-BA)$

5. (1) 设 $k_1 T a_1 + k_2 T a_2 + \cdots + k_n T a_n = 0$

则 $\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = 0$ 系数矩阵为Vandermonde矩阵

行列式非0

由Cramer法则知 $k_1 = k_2 = \cdots = k_n = 0$

故 $Ta_1, \cdots, Ta_n$ 线性无关

(2) 设对偶基为 $f_1, \cdots, f_n$

则 $f_i(a_j) = \delta_{ij} \Rightarrow (x-a_j) \mid f_i \ (i \neq j)$ 由 a_i 互不相同知

$\prod_{j\neq i}(x-a_j) \mid f_i$, 又 $f_i \in \mathbb{R}[x]^{(n)} \Rightarrow f_i \sim \prod_{j\neq i}(x-a_j)$

$f_i(a_i) = 1 \Rightarrow f_i = \dfrac{\prod_{j\neq i}(x-a_j)}{\prod_{j\neq i}(a_i-a_j)}$

(3) 记 $P_k(x_1, \cdots, x_{n-1}) = \sum_{1\leq i_1<\cdots<i_k\leq n-1} x_{i_1}x_{i_2}\cdots x_{i_k} \quad (1\leq k\leq n-1)$

则由Vieta定理知

$f_i = \dfrac{1}{\prod_{j\neq i}(a_i-a_j)}\left(P_0(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)x^{n-1} + \cdots + P_{n-1}(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)\right)$

故转换矩阵 i 行 j 列元为 $\dfrac{1}{\prod_{k\neq j}(a_j-a_k)} P_{n-i}(a_1,\cdots,a_{j-1},a_{j+1},\cdots,a_n)$

2、极化公式

(1) V上双线性型f反对称 $\Leftrightarrow \forall x\in V, f(x,x)=0$ $(\mathrm{char}\,\mathbb{F}\neq 2)$

(2) $Q: V\to \mathbb{Q}$, $f=\frac{1}{2}(Q(\alpha+\beta)-Q(\alpha)-Q(\beta))$, V为$\mathbb{Q}$上线性空间

则f为对称双线性函数 $\Leftrightarrow Q(k,\alpha)=k^2\alpha$, $Q(\alpha+\beta)+Q(\alpha-\beta)=2(Q(\alpha)-Q(\beta))$ $(\forall k\in\mathbb{Q}, \alpha,\beta\in V)$

若Q连续，则$\mathbb{Q}$可改为$\mathbb{R}$.

(1) $f(x,y)=-f(y,x)\Rightarrow f(x,x)=-f(x,x)\Rightarrow f(x,x)=0$

反之，$\forall x,y\in V$, $f(x+y,x+y)-f(x,x)-f(y,y)=f(x,y)+f(y,x)$

$\Rightarrow f(x,y)=-f(y,x)$

(2) "$\Rightarrow$": 略

"$\Leftarrow$": $\forall \alpha,\beta,\gamma\in V$　易知f对称

$f(\alpha+\beta,\gamma)=f(\alpha,\gamma)+f(\beta,\gamma)$

$\Leftrightarrow Q(\alpha+\beta+\gamma)-Q(\alpha+\beta)=Q(\alpha+\gamma)-Q(\alpha)+Q(\beta+\gamma)-Q(\beta)-Q(\gamma)$

$\Leftrightarrow Q(\alpha+\beta+\gamma)-Q(\alpha+\beta)=\frac{1}{2}(Q(\alpha+\beta+2\gamma)+Q(\alpha-\beta))-Q(\alpha)-Q(\beta)-Q(\gamma)$

$\Leftrightarrow Q(\alpha+\beta+\gamma)-Q(\alpha+\beta)=Q(\alpha+\beta+\gamma)+Q(\gamma)-\frac{Q(\alpha+\beta)}{2}+\frac{Q(\alpha-\beta)}{2}-Q(\alpha)-Q(\beta)-Q(\gamma)$

$\Leftrightarrow -Q(\alpha+\beta)=Q(\alpha-\beta)-2(Q(\alpha)+Q(\beta))$

故上式成立. $\forall \frac{q}{p}\in\mathbb{Q}$　由 $f(p\cdot\frac{q}{p}\alpha,\beta)=f(q\alpha,\beta)\Rightarrow f(\frac{q}{p}\alpha,\beta)=\frac{1}{p}f(\alpha,\beta)$

知 $pf(\frac{q}{p}\alpha,\beta)=f(q\alpha,\beta)=qf(\alpha,\beta)\Rightarrow f(\frac{q}{p}\alpha,\beta)=\frac{q}{p}f(\alpha,\beta)$

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若 $\mathbb{Q}$ 连续, 则由 $\mathbb{Q}$ 在 $\mathbb{R}$ 稠密知 f 为 $\mathbb{R}$-线性空间 V 上双线性函数

3、$A \sim_c B$ 且 A, B 可逆 则 $A^{-1} \sim_c B^{-1}$

$A \sim_c B$, $C \sim_c D$, 则 $\begin{pmatrix} A & \\ & C \end{pmatrix} \sim_c \begin{pmatrix} B & \\ & D \end{pmatrix}$

证明略

化实二次型为规范型

(1) $f = x_1 x_2 + x_2 x_3$

(2) $f = (x_1 - a x_2)^2 + \cdots + (x_{n-1} - a x_n)^2 + (x_n - \alpha x_1)^2$

(1) $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ 则 $y_1^2 - y_2^2 = (x_1 + x_3) x_2 = x_1 x_2 + x_2 x_3 = f$

故规范型为 $y_1^2 - y_2^2$

(2) 记 $A = \begin{pmatrix} 1 & -\alpha & & & \\ & 1 & -\alpha & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha \\ -\alpha & & & & 1 \end{pmatrix}$ $\det A = 1 + (-1)^{n+1}(-\alpha)(-\alpha)^{n-1} = 1 - \alpha^n$

则 $f(X) = (AX)^t AX = X^t A^t AX$

$\alpha^n \neq 1$ 时 $\det A \neq 0$, $A^t A \sim_c I$ 即规范型 $y_1^2 + \cdots + y_n^2$

$\alpha^n = 1$ 时 $\exists Q \in GL_n(\mathbb{R})$ s.t. $AQ = \begin{pmatrix} I_{n-1} & 0 \\ \alpha^t & 0 \end{pmatrix}$ $(\alpha \neq 0)$

故 $A^t A \sim_c Q^t A^t A Q = \begin{pmatrix} I_{n-1} + \alpha \alpha^t & 0 \\ 0 & 0 \end{pmatrix}$

取 $\alpha^t X = 0$ 的解空间 W_1 中元 ω_1, 取 $\begin{pmatrix} \alpha^t \\ \omega_1^t \end{pmatrix} X = 0$ 解空间 W_2 中元 ω_2

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$(w_i \neq 0)$ 重复此过程得到 W_1 的基 $w_1, \cdots, w_{n-1}$ s.t. $w_i^t w_j = 0$ $(i \neq j)$

取 $P = (\alpha, w_1, \cdots, w_{n-1})$ 则 s.t. $P^t \alpha \alpha^t P = (\alpha^t \alpha)^2 E_{11}$

且 $P^t E P = diag(\alpha^t \alpha, w_1^t w_1, \cdots, w_{n-1}^t w_{n-1})$ 由 $w_i^t w_i > 0, \alpha^t \alpha > 0$

知有规范型 $y_1^2 + \cdots + y_{n-1}^2$

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