1. 设 W 是

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

在 $\mathbb{R}^4$ 中的解空间. 求 $W^\perp$ 的一组单位正交基.

解: 记 $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$ 则

$$A \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而 $\mathrm{rank}(A)=2$

$\dim(W^\perp) = 4 - \dim(W) = 4 - \mathrm{rank}(\mathrm{sol}(Ax=0))$
$= \mathrm{rank}(A) = 2$

记 $\vec{A}_i$ 为矩阵 A 的第 i 行, $\because \langle \vec{A}_1^{\,t}, \vec{A}_2^{\,t}, \vec{A}_3^{\,t}, \vec{A}_4^{\,t} \rangle \subseteq W^\perp$

$\therefore \langle \vec{A}_1^{\,t}, \vec{A}_2^{\,t}, \vec{A}_3^{\,t}, \vec{A}_4^{\,t} \rangle = W^\perp$ $\therefore$

$\because \vec{A}_1^{\,t}$ 与 $\vec{A}_2^{\,t}$ 线性无关 且 $\dim(W^\perp)=2$, 所以 $\{\vec{A}_1^{\,t}, \vec{A}_2^{\,t}\}$ 为 $W^\perp$ 的一组基.

下面对 $\vec{A}_1^{\,t}, \vec{A}_2^{\,t}$ 做 Gram-Schmidt 正交化

$$\vec{e}_1 = \frac{\vec{A}_1^{\,t}}{\|\vec{A}_1^{\,t}\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\vec{e}_2{}' = \vec{A}_2^{\,t} - (\vec{A}_2^{\,t}, \vec{e}_1)\vec{e}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ \frac{3}{2} \end{pmatrix}$$

$$\vec{e}_2 = \frac{\vec{e}_2{}'}{\|\vec{e}_2{}'\|} = \frac{\vec{e}_2{}'}{\frac{\sqrt{22}}{2}} = \frac{2\sqrt{22}}{22} \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{22}}{22} \\ \frac{2\sqrt{22}}{22} \\ 0 \\ \frac{3\sqrt{22}}{22} \end{pmatrix}$$

$\square$

2. 设 V 是欧式空间, $\mathbf{v}$ 是 V 的单位向量. 设

$$\begin{aligned}\mathcal{A}:\ V &\longrightarrow V\\ \mathbf{x} &\mapsto 2(\mathbf{x}|\mathbf{v})\mathbf{v}-\mathbf{x}.\end{aligned}$$

(i) 证明: $\mathcal{A}$ 是线性算子.

(ii) 证明: $\mathcal{A}$ 既是对称算子又是正交算子.

(iii) 计算 $\mathcal{A}$ 的所有特征子空间的维数.

证:(i) 这样的映射称为一个镜面反射

$\forall \vec{x}_1,\vec{x}_2\in V,\quad \alpha,\beta\in\mathbb{R}\quad \mathcal{A}(\alpha\vec{x}_1+\beta\vec{x}_2)=2(\alpha\vec{x}_1+\beta\vec{x}_2|\vec{v})\vec{v}-(\alpha\vec{x}_1+\beta\vec{x}_2)$

$=2\left(\alpha(\vec{x}_1|\vec{v})+\beta(\vec{x}_2|\vec{v})\right)\vec{v}-(\alpha\vec{x}_1+\beta\vec{x}_2)$

$=2\alpha(\vec{x}_1|\vec{v})\vec{v}+2\beta(\vec{x}_2|\vec{v})\vec{v}-(\alpha\vec{x}_1+\beta\vec{x}_2)$

$=2\alpha(\vec{x}_1|\vec{v})\vec{v}-\alpha\vec{x}_1+2\beta(\vec{x}_2|\vec{v})\vec{v}-\beta\vec{x}_2$

$=2(\alpha\vec{x}_1|\vec{v})\vec{v}-\alpha\vec{x}_1+2(\beta\vec{x}_2|\vec{v})\vec{v}-\beta\vec{x}_2$

$=\alpha\mathcal{A}(\vec{x}_1)+\beta\mathcal{A}(\vec{x}_2)\quad\therefore\ \mathcal{A}$ 是线性算子.

法二: 因为 ε 和 $\mathcal{B}:=2(\cdot|\vec{v})\vec{v}$ 都是线性算子 且 $\mathcal{A}=\mathcal{B}-\varepsilon\ \therefore$

$\mathcal{A}$ 是线性算子.

(ii) 对称算子: $\forall x,y\in V\quad(\mathcal{A}(\vec{x})|\vec{y})=(\vec{x}|\mathcal{A}(\vec{y}))$

正交算子: 保内(长)算子: $\forall x\in V,\ \|x\|=\|\mathcal{A}(x)\|$

$(\mathcal{A}(\vec{x})|\vec{y})=\left(2(\vec{x}|\vec{v})\vec{v}-\vec{x}|\vec{y}\right)=2\left((\vec{x}|\vec{v})\vec{v}|\vec{y}\right)-(\vec{x}|\vec{y})$

$=2(\vec{x}|\vec{v})(\vec{v}|\vec{y})-(\vec{x}|\vec{y})$

$=2(\vec{v}|\vec{y})\cdot(\vec{x}|\vec{v})-(\vec{x}|\vec{y})$ (x不动, 利用内积关于第二个变元的线性性)

$= (\vec{x} \mid 2(\vec{v}\mid\vec{y})\cdot\vec{v} - \vec{y})$

其中 $2(\vec{v}\mid\vec{y})\cdot\vec{v} - \vec{y} = \mathcal{A}(\vec{y})$

下证对 $\forall \vec{x} \in V$，有 $\|\vec{x}\| = \|\mathcal{A}(\vec{x})\|$

$(\mathcal{A}(\vec{x}) \mid \mathcal{A}(\vec{x})) = (2(\vec{x}\mid\vec{v})\vec{v} - \vec{x},\ 2(\vec{x}\mid\vec{v})\vec{v} - \vec{x})$

$= 4(\vec{x}\mid\vec{v})^2\cdot(\vec{v},\vec{v}) - 2(\vec{x}\mid\vec{v})(\vec{v},\vec{x}) - 2(\vec{x}\mid\vec{v})(\vec{x}\mid\vec{v}) + (\vec{x},\vec{x})$

$\because$ $\vec{v}$ 单位向量 $(\vec{v},\vec{v}) = \|\vec{v}\|^2 = 1$

$\therefore$ $= 4(\vec{x}\mid\vec{v})^2 - 2(\vec{x}\mid\vec{v})^2 - 2(\vec{x}\mid\vec{v})^2 + (\vec{x},\vec{x})$

$= (\vec{x}\mid\vec{x})$

或直接由 $\mathcal{A}$ 既对称又正交 $\therefore$ $\begin{cases} AA^t = E \\ A = A^t \end{cases} \Rightarrow A^2 = E$

(iii) $\mathcal{A}^2(\vec{x}) = \mathcal{A}(2(\vec{x}\mid\vec{v})\vec{v} - \vec{x}) = 2(\vec{x}\mid\vec{v})\,\mathcal{A}(\vec{v}) - \mathcal{A}(\vec{x})$

$= 2(\vec{x}\mid\vec{v})\cdot(2(\vec{v}\mid\vec{v})\vec{v} - \vec{v}) - 2(\vec{x}\mid\vec{v})\vec{v} + \vec{x}$

$= 2(\vec{x}\mid\vec{v})\vec{v} - 2(\vec{x}\mid\vec{v})\vec{v} + \vec{x} = \vec{x}$

$\because$ $\mathcal{A} + \mathcal{E} \neq \mathcal{O}$ 且 $\mathcal{A} - \mathcal{E} \neq \mathcal{O}$ $\therefore$ $\mathcal{A}$ 的极小多项式为 $t^2 - 1$

从而 $\mathcal{A}$ 的特征值为 1 和 -1

$\vec{x} \in V^1 \Leftrightarrow \mathcal{A}(\vec{x}) = \vec{x} \Longleftrightarrow 2(\vec{x}\mid\vec{v})\vec{v} - \vec{x} = \vec{x}$

$\Leftrightarrow \vec{x} = (\vec{x}\mid\vec{v})\vec{v} \Leftrightarrow \vec{x} \in \langle \vec{v} \rangle$

$\therefore V^1 = \langle \vec{v} \rangle$

$\vec{x} \in V^{-1} \Leftrightarrow \mathcal{A}(\vec{x}) = -\vec{x} \Longleftrightarrow 2(\vec{x}\mid\vec{v})\vec{v} - \vec{x} = -\vec{x}$

$\Leftrightarrow 2(\vec{x}\mid\vec{v})\vec{v} = \vec{0} \Longleftrightarrow (\vec{x}\mid\vec{v}) = \vec{0} \Leftrightarrow \vec{x} \in \langle \vec{v} \rangle^{\perp}$

$\Rightarrow V^{-1} = \langle \vec{v} \rangle^{\perp}$

从而 $\dim(V^1)=1$, $\dim(V^{-1})=n-1$.

$\square$

法二：$\vec{v}$ 是单位向量，设 $\{\vec{v}, \vec{e}_2, \cdots, \vec{e}_n\}$ 是 V 的一组单位正交基.

则 $\mathcal{A}(\vec{v})=\vec{v}$ 且 $\mathcal{A}(\vec{e}_i)=2(\vec{e}_i|\vec{v})\vec{v}-\vec{e}_i=-\vec{e}_i$ $\forall i\in\{2,\cdots,n\}$

所以 $\mathcal{A}$ 在这组基下的矩阵为

$$\mathcal{A}(\vec{v}, \vec{e}_2, \cdots, \vec{e}_n)=(\vec{v}, \vec{e}_2, \cdots, \vec{e}_n)\cdot\underbrace{\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}}_{A}$$

$\chi_A=|tE-A|=(t-1)(t+1)^{n-1}$

$\Rightarrow$ 特征值为 1 和 -1

$\dim(V^1)=n-\operatorname{rank}(1\cdot E-A)=1$

$\dim(V^{-1})=n-\operatorname{rank}(-1\cdot E-A)=n-1$

3. (i) 设 U_1, U_2 都是 V 的子空间, 且 $U_1 \subset U_2$. 证明: $U_2^{\perp} \subset U_1^{\perp}$;

(ii) 设 U_1, U_2 都是 V 的子空间. 证明: $(U_1 \cap U_2)^{\perp} = U_1^{\perp} + U_2^{\perp}$.

证. (i) $\forall \vec{x} \in U_2^{\perp}$, 对 $\forall \vec{y} \in U_2$ 有 $\vec{x} \perp \vec{y}$. 从而对 $\forall \vec{y} \in U_1$, $\vec{x} \perp \vec{y}$

$\therefore \vec{x} \in U_1^{\perp} \Rightarrow U_2^{\perp} \subseteq U_1^{\perp}$

(ii) $U_1 \cap U_2 \subseteq U_1$ 且 $U_1 \cap U_2 \subseteq U_2$

$\Rightarrow U_1^{\perp} \subseteq (U_1 \cap U_2)^{\perp}$ 且 $U_2^{\perp} \subseteq (U_1 \cap U_2)^{\perp}$

$\therefore U_1^{\perp} + U_2^{\perp} \subseteq (U_1 \cap U_2)^{\perp}$

反过来: 要证 $(U_1 \cap U_2)^{\perp} \subseteq U_1^{\perp} + U_2^{\perp}$

只需证 $(U_1^{\perp} + U_2^{\perp})^{\perp} \subseteq U_1 \cap U_2$

对 $\forall \vec{y} \in (U_1^{\perp} + U_2^{\perp})^{\perp}$ 对 $\forall \vec{x} \in U_1^{\perp} + U_2^{\perp}$ 有 $\vec{y} \perp \vec{x}$

$\because U_1^{\perp} \subseteq U_1^{\perp} + U_2^{\perp}$ 所以对 $\forall \vec{x} \in U_1^{\perp}$ 有 $\vec{y} \perp \vec{x}$

$\Rightarrow \vec{y} \in (U_1^{\perp})^{\perp} \Rightarrow \vec{y} \in U_1$

同理 $\because U_2^{\perp} \subseteq U_1^{\perp} + U_2^{\perp}$ 所以对 $\forall \vec{x} \in U_2^{\perp}$ 有 $\vec{y} \perp \vec{x}$

$\Rightarrow \vec{y} \in (U_2^{\perp})^{\perp} \Rightarrow \vec{y} \in U_2 \Rightarrow \vec{y} \in U_1 \cap U_2$ $\square$

4. 设 $\mathrm{O}_n(\mathbb{R})$ 是所有 n 阶正交矩阵关于乘法构成的群.

(i) 令 $\mathrm{SO}_n(\mathbb{R})=\{P\in \mathrm{O}_n(\mathbb{R})|\det(P)=1\}$. 证明: $\mathrm{SO}_n(\mathbb{R})$ 是 $\mathrm{O}_n(\mathbb{R})$ 的子群;

(ii) 设 $T\in \mathrm{O}_n(\mathbb{R})$ 是上三角的. 证明: T 是对角的.

证: (i) 首先由 $SO_n(\mathbb{R})$ 的定义知 $SO_n(\mathbb{R})\subseteq O_n(\mathbb{R})$

且 $\forall\, P, Q\in SO_n(\mathbb{R})$, $\Rightarrow P, Q\in O_n(\mathbb{R})$.

$\because O_n(\mathbb{R})$ 是群, $\therefore PQ^{-1}\in O_n(\mathbb{R})$

$\det(PQ^{-1})=\det(P)\cdot\det(Q^{-1})=1\cdot 1=1$

$\Rightarrow PQ^{-1}\in SO_n(\mathbb{R})$. 从而 $SO_n(\mathbb{R})$ 是 $O_n(\mathbb{R})$ 子群.

(ii) $T^t=T^{-1}\Rightarrow T\cdot T^t=E$

设 $T=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn}\end{pmatrix}$ 则 $T^t=\begin{pmatrix} a_{11} & & & 0 \\ a_{12} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{pmatrix}$

$\Rightarrow T\cdot T^t=\begin{pmatrix} a_{11}^2+\cdots+a_{1n}^2 & & & * \\ & a_{22}^2+\cdots+a_{2n}^2 & & \\ & & \ddots & \\ * & & & a_{nn}^2\end{pmatrix}=E$

且 $T^t\cdot T=\begin{pmatrix} a_{11}^2 & & & & * \\ & a_{12}^2+a_{22}^2 & & & \\ & & a_{13}^2+\cdots+a_{33}^2 & & \\ & & & \ddots & \\ * & & & & a_{1n}^2+a_{2n}^2+\cdots+a_{nn}^2\end{pmatrix}=E$

$\Rightarrow \begin{cases} a_{11}^2+\cdots+a_{1n}^2=1 \\ a_{22}^2+\cdots+a_{2n}^2=1 \\ a_{33}^2+\cdots+a_{3n}^2=1 \\ \vdots \\ a_{nn}^2=1\end{cases}$ 且 $\begin{cases} a_{11}^2=1 \\ a_{12}^2+a_{22}^2=1 \\ a_{13}^2+\cdots+a_{33}^2=1 \\ \vdots \\ a_{1n}^2+\cdots+a_{nn}^2=1\end{cases}$

$\Rightarrow a_{12}=\cdots=a_{1n}=0,\quad a_{23}=\cdots=a_{2n}=0,\quad a_{34}=\cdots=a_{3n}=0$

$,\cdots\quad a_{n-1\,n}=0$

从而T为对角矩阵.

或者 $\because T\in O_n(\mathbb{R})\ \therefore T$ 的列组成一组单位正交基.

$\therefore T$ 的每一列都是单位向量.

记 $\vec{T}^{(i)}$ 为T的第i列

$\|\vec{T}^{(1)}\|=1 \Rightarrow a_{11}=\pm1$

由 $(\vec{T}^{(1)},\vec{T}^{(2)})=0 \Rightarrow a_{11}\cdot a_{12}=0 \underset{a_{11}=\pm1}{\Rightarrow} a_{12}=0 \xrightarrow{\vec{T}^{(2)}\text{单位}} a_{22}=\pm1$

由 $(\vec{T}^{(1)},\vec{T}^{(3)})=0 \Rightarrow a_{11}\cdot a_{13}=0 \underset{a_{11}=\pm1}{\Rightarrow} a_{13}=0$

同理可推出 $a_{1i}=0\quad \forall 2\le i\le n$.

由 $(\vec{T}^{(2)},\vec{T}^{(3)})=0 \Rightarrow a_{22}\cdot a_{23}=0 \Rightarrow a_{23}=0 \xrightarrow{\vec{T}^{(3)}\text{单位}} a_{33}=\pm1$

$\vdots$

$\Rightarrow a_{2i}=0\quad \forall 3\le i\le n$

由数学归纳法可得T为对角矩阵. □

法二：对 n 归纳 $n=1$, ✓ 设 $n>1$ 且 $n-1$ 时结论成立

因为 T 是上三角矩阵, ∴ $T=\begin{pmatrix} a & \vec{c} \\ 0 & B \end{pmatrix}$ 其中 $a\in\mathbb{R}$, $\vec{c}\in\mathbb{R}^{1\times n-1}$, $B\in M_{n-1}(\mathbb{R})$

且 B 是上三角矩阵, T 正交, 从而一定为正规矩阵. ∴ 由引理 4.11

知 $\vec{c}=\vec{0}$. (引理 4.11: $A=\begin{pmatrix} B & C \\ O_{(n-k)\times k} & D \end{pmatrix}$, $B\in M_k(\mathbb{R})$ 若 A 正规, 则 $C=O_{k\times(n-k)}$)

∵ $T\in O_n(\mathbb{R})$ ∴ $T\cdot T^t=E$

$$\Rightarrow \begin{pmatrix} a & \vec{c} \\ 0 & B \end{pmatrix}\begin{pmatrix} a & 0 \\ \vec{c}^{\,t} & B^t \end{pmatrix}=E \Rightarrow \begin{pmatrix} a^2 & \vec{0} \\ \vec{0} & B\cdot B^t \end{pmatrix}=E$$

$\Rightarrow a=\pm1$, 且 $B\cdot B^t=E_{n-1} \Rightarrow B$ 正交, 由归纳假设知 B 为对角阵.

□

5. 设 V 是 n 维欧氏空间, $\mathbf{e}_1,\dots,\mathbf{e}_n\in V$. 证明下列断言等价:

(i) $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ 是 V 的一组单位正交基;

(ii) 对任意的 $\mathbf{x},\mathbf{y}\in V, (\mathbf{x}\mid\mathbf{y})=\sum_{i=1}^n(\mathbf{x}\mid\mathbf{e}_i)(\mathbf{y}\mid\mathbf{e}_i)$;

(iii) 对任意的 $\mathbf{x}\in V, \|\mathbf{x}\|^2=\sum_{i=1}^n(\mathbf{x}\mid\mathbf{e}_i)^2$.

证: (i)⇒(ii) $\{\vec{e}_1,\dots,\vec{e}_n\}$ 是一组单位正交基, ∴ $(\vec{e}_i|\vec{e}_j)=0$, $(\vec{e}_i|\vec{e}_i)=1$

且 $\forall\,\vec{x},\vec{y}\in V$, $\vec{x}=\sum_{i=1}^n(\vec{x}|\vec{e}_i)\vec{e}_i$ $\vec{y}=\sum_{i=1}^n(\vec{y}|\vec{e}_i)\vec{e}_i$

$$\therefore (\vec{x}|\vec{y}) = \left(\sum_{i=1}^{n}(\vec{x}|\vec{e_i})\vec{e_i} \,\middle|\, \sum_{i=1}^{n}(\vec{y}|\vec{e_i})\vec{e_i}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}(\vec{x}|\vec{e_i})(\vec{y}|\vec{e_j})(\vec{e_i}|\vec{e_j})$$

$$= \sum_{i=1}^{n}(\vec{x}|\vec{e_i})(\vec{y}|\vec{e_i})$$

(ii)$\Rightarrow$(iii) $\|x\|^2 = (x|x) = \sum_{i=1}^{n}(x|e_i)^2$

(iii)$\Rightarrow$(i) 取 $\vec{x} = \vec{e_1}$ 则 $\|\vec{e_1}\|^2 = \sum_{i=1}^{n}(e_1|e_i)^2 = \|e_1\|^4 + \sum_{i=2}^{n}(e_1|e_i)^2$ (※)

$\Rightarrow \|\vec{e_1}\|^4 \le \|\vec{e_1}\|^2 \Rightarrow \|\vec{e_1}\|^2 \le 1 \Rightarrow \|\vec{e_1}\| \le 1$

$\because V = \underbrace{\langle \vec{e_2}, \cdots \vec{e_n} \rangle}_{\dim \le n-2} \oplus \underbrace{\langle \vec{e_2} \cdots, \vec{e_n} \rangle^{\perp}}_{\dim \ge 2}$

$\therefore \langle \vec{e_2} \cdots \vec{e_n} \rangle^{\perp} \ne \{0\}$ 取 $\vec{w} \in \langle \vec{e_2} \cdots \vec{e_n} \rangle^{\perp} \setminus \{0\}$

则 $\|\vec{w}\| = \sum_{i=1}^{n}(\vec{w}|\vec{e_i})^2 = (\vec{w}|\vec{e_1})^2 \le \|\vec{w}\| \cdot \|\vec{e_1}\|$

$\Rightarrow \|\vec{e_1}\| \ge 1 \Rightarrow \vec{e_1}$ 是单位向量

且由(※)知 $1 = 1 + \sum_{i=2}^{n}(e_1|e_i)^2 \Rightarrow \sum_{i=2}^{n}(e_1|e_i)^2 = 0$

$\Rightarrow e_1 \perp e_i \quad \forall i \in \{2, \cdots, n\}$

同理可证对 $\forall i \in \{2, \cdots, n\}$ $\vec{e_i}$ 都是单位向量且与 $\vec{e_1} \cdots \vec{e_{i-1}}, \vec{e_{i+1}} \cdots \vec{e_n}$ 正交

$\square$