1. 设实对称矩阵

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \quad 和 \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) 求正交矩阵 P, Q 与对角阵 C, D 使得 $P^t AP = C$ 和 $Q^t BQ = D$.

(b) 计算 A^k, B^k, $k \in \mathbb{Z}^+$.

解：问题：设 $A \in SM_n(\mathbb{R})$，求 $P \in O_n(\mathbb{R})$ 使得

$$P^t A P = diag(\lambda_1, \cdots, \lambda_n)$$

基本步骤：1. 计算 A 的特征根，设互不相同的特征根是 $\lambda_1, \cdots, \lambda_k$；

2. 对 $i \in \{1, 2, \cdots, k\}$，求 V^{λ_i} 的一组基；

3. 对 $i \in \{1, 2, \cdots, k\}$，利用 Gram-Schmidt 正交化求 V^{λ_i} 的一组单位正交基：$\vec{e}_{i,1}, \cdots, \vec{e}_{i,d_i}$

4. $P = (\vec{e}_{1,1}, \cdots \vec{e}_{1,d_1} \cdots \vec{e}_{k,1} \cdots \vec{e}_{k,d_k})$

则 $P^t A P =$ 对角矩阵

(a)：$|tE - A| = 0$

$$\Rightarrow \begin{vmatrix} t-\cos\theta & -\sin\theta \\ -\sin\theta & t+\cos\theta \end{vmatrix} = 0 \Rightarrow t^2 - \cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow t^2 - 1 = 0$$

① $\Rightarrow t = \pm 1$，令 $\lambda_1 = 1$，$\lambda_2 = -1$ $\Rightarrow \dim V^{\lambda_1} = \dim V^{\lambda_2} = 1$

当特征值为1时 $(1 \cdot E - A)\vec{x} = \vec{0}$

$$\begin{pmatrix} 1-\cos\theta & -\sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

② 只需考虑 $(1-\cos\theta)x_1 - \sin\theta\, x_2 = 0 \Rightarrow V^{\lambda_1} = \left\langle \begin{pmatrix} \sin\theta \\ 1-\cos\theta \end{pmatrix} \right\rangle$

当特征值为-1时 $(-E-A)\vec{x} = \vec{0}$

$$\Rightarrow \begin{pmatrix} -1-\cos\theta & -\sin\theta \\ -\sin\theta & -1+\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

只需考虑 $(1+\cos\theta)x_1+\sin\theta x_2=0 \Rightarrow V^{\lambda_2}=<\begin{pmatrix}-\sin\theta\\ 1+\cos\theta\end{pmatrix}>$

或者利用 $V^{\lambda_2}=(V^{\lambda_1})^{\perp} \Rightarrow V^{\lambda_2}=<\begin{pmatrix}1-\cos\theta\\ -\sin\theta\end{pmatrix}>$

③ $\sqrt{\sin^2\theta+(1-\cos\theta)^2}=\sqrt{2-2\cos\theta}$ 当 $\cos\theta=1$ 时

$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$，A为对角矩阵 $\therefore P=E_2$

$\sqrt{\sin^2\theta+(1+\cos\theta)^2}=\sqrt{2+2\cos\theta}$ 当 $\cos\theta=-1$ 时

$A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ A为对角矩阵 $\therefore P=E_2$

当 $\cos\theta\neq\pm1$ 时，$\varepsilon_1=\dfrac{1}{\sqrt{2-2\cos\theta}}\cdot\begin{pmatrix}\sin\theta\\ 1-\cos\theta\end{pmatrix}$

$1+\cos\theta=2\cos^2\frac{\theta}{2}$

$1-\cos\theta=2\sin^2\frac{\theta}{2}$

$$\varepsilon_2=\frac{1}{\sqrt{2+2\cos\theta}}\begin{pmatrix}-\sin\theta\\ 1+\cos\theta\end{pmatrix}$$

或者不用讨论 $\cos\theta$ 的值，$\varepsilon_1=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$ $\varepsilon_2=\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$

为满足条件的两个单位特征向量.

$$\therefore P=\begin{cases}\begin{pmatrix}\dfrac{\sin\theta}{\sqrt{2-2\cos\theta}} & \dfrac{-\sin\theta}{\sqrt{2+2\cos\theta}}\\ \dfrac{1-\cos\theta}{\sqrt{2-2\cos\theta}} & \dfrac{1+\cos\theta}{\sqrt{2+2\cos\theta}}\end{pmatrix} & \cos\theta\neq\pm1\\ E_2 & \cos\theta=\pm1\end{cases}$$

式 $P = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

对矩阵 B $\det(tE - B) = (t-1)^2(t-4)$

$\lambda_1 = 1$ $\lambda_2 = 4.$ $\dim(V^{\lambda_1}) = 2$ $\dim(V^{\lambda_2}) = 1$

$\Rightarrow \mathrm{rank}(1\cdot E - B) = 1$ $\Rightarrow \mathrm{rank}(4\cdot E - B) = 2$

$-x_1 - x_2 - x_3 = 0 \Rightarrow V^{\lambda_1} = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\because V^{\lambda_2} \perp V^{\lambda_1}$ 所以 $V^{\lambda_2} = \left\langle \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\rangle$

对 V^{λ_1} 中的向量做 G-S 单位正交化

$\vec{\varepsilon}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{\varepsilon}_2{}' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$\vec{\varepsilon}_2 = \frac{1}{\frac{\sqrt{6}}{2}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ $\vec{\varepsilon}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

所以 $Q = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(b) $(P^tAP)^k = P^tA^kP = C^k = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix}$

所以 $A^k = PC^kP^t = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$

$$
= \begin{pmatrix} \cos^2\frac{\theta}{2} + (-1)^k \sin^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} + (-1)^{k+1}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} + (-1)^{k+1}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} + (-1)^k\cos^2\frac{\theta}{2} \end{pmatrix}
$$

$$
= \begin{cases} A & k\text{奇} \\ E & k\text{偶} \end{cases}
$$

同理 $(Q^t B Q)^k = Q^t B^k Q = D^k = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 4^k \end{pmatrix}$

$$
\Longrightarrow B^k = Q D^k Q^t = \begin{pmatrix} \frac{4^k+2}{3} & \frac{4^k-1}{3} & \frac{4^k-1}{3} \\ \frac{4^k-1}{3} & \frac{4^k+2}{3} & \frac{4^k-1}{3} \\ \frac{4^k-1}{3} & \frac{4^k-1}{3} & \frac{4^k+2}{3} \end{pmatrix}
$$

$\square$

另种求 C^n 的方法: Hamilton-Cayley 定理:

设 $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$ 求 $\begin{pmatrix} A & \\ & B \end{pmatrix}^n$ 对 $\forall n \in \mathbb{N}^+$

解: $\chi_A = \begin{vmatrix} t & -1 \\ -1 & t \end{vmatrix} = t^2 - 1$ $\chi_B = \begin{vmatrix} t & 2 \\ -1 & t \end{vmatrix} = t^2 + 2$

也可验证 t^2-1 为 A 的极小多项式 且 t^2+2 是 B 的极小多项式.

$\therefore A^2 = E$ $B^2 = -2E$

n 为偶时 $A^n = A^2 A^{n-2} = A^{n-2} = A^2 A^{n-4} = \cdots = E$

$B^n = (-2)^{\frac{n}{2}} \cdot E$

n 为奇时 $A^n = A \cdot A^{n-1} = A$

$$B^n = B\cdot(-2)^{\frac{n-1}{2}}$$

所以 $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^n = \begin{pmatrix} A^n & 0 \\ 0 & B^n \end{pmatrix} = \begin{cases} \begin{pmatrix} E_2 & (-2)^{\frac{n}{2}} E_2 \end{pmatrix} & n\text{偶} \\ \begin{pmatrix} A & (-2)^{\frac{n-1}{2}} B \end{pmatrix} & n\text{奇}. \end{cases}$

(b) 计算 $A^n, B^n, n \in \mathbb{Z}^+$.

2. 设 $A, B \in \mathrm{SM}_n(\mathbb{R})$. $\lambda_1 \leq \cdots \leq \lambda_n$ 为 A 的所有特征值, $\mu_1 \leq \cdots \leq \mu_n$ 为 B 的所有特征值, 证明:

(a) $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$, 定义

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^t A \mathbf{x}}{\mathbf{x}^t \mathbf{x}},$$

则 $\min(R_A(\mathbf{x})) = \lambda_1$ 和 $\max(R_A(\mathbf{x})) = \lambda_n$.

(b) $A + B$ 的特征值均落在 $[\lambda_1 + \mu_1, \lambda_n + \mu_n]$ 之中.

证: $\because A \in SM_n(\mathbb{R})$ $\therefore \exists P \in O_n(\mathbb{R})$ s.t.

$P^t A P = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) := C$

从而 $A = P C P^t$

所以 $\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$,

$$R_A(\vec{x}) = \frac{\vec{x}^t A \vec{x}}{\vec{x}^t \vec{x}} = \frac{\vec{x}^t P C P^t \vec{x}}{\vec{x}^t \vec{x}} = \frac{(P^t \vec{x})^t C (P^t \vec{x})}{\vec{x}^t \vec{x}}$$

又∵ $(P^t\vec{x})^t\cdot(P^t\vec{x}) = \vec{x}^t \underbrace{P\cdot P^t}_{正交}\vec{x} = \vec{x}^t\cdot\vec{x}$

$$\therefore\ R_A\vec{x} = \frac{(P^t\vec{x})^t\, C\,(P^t\vec{x})}{(P^t\vec{x})^t\cdot(P^t\vec{x})}$$

令 $\vec{y} = P^t\vec{x}$ 则 $R_A\vec{x} = \dfrac{\vec{y}^t C\vec{y}}{\vec{y}^t\vec{y}} = \dfrac{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}{y_1^2 + \cdots + y_n^2}$

$\lambda_1(y_1^2+\cdots+y_n^2) \le \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \le \lambda_n(y_1^2+\cdots+y_n^2)$

所以 $\lambda_1 \le R_A\vec{x} \le \lambda_n$

并且 $R_A\vec{x} = \lambda_1 \iff y_1 \ne 0,\ y_2 = \cdots = y_n = 0$

$R_A\vec{x} = \lambda_n \iff y_n \ne 0,\ y_1 = \cdots = y_{n-1} = 0$

$\therefore\ \min R_A\vec{x} = \lambda_1, \quad \max R_A\vec{x} = \lambda_n$

(b): $R_{A+B}(\vec{x}) = \dfrac{\vec{x}^t(A+B)\vec{x}}{\vec{x}^t\vec{x}} = \dfrac{\vec{x}^t A\vec{x}}{\vec{x}^t\vec{x}} + \dfrac{\vec{x}^t B\vec{x}}{\vec{x}^t\vec{x}}$

$= R_A(\vec{x}) + R_B(\vec{x})$

$\min R_{A+B}(\vec{x}) = \min\{A+B$ 的特征值$\} := \alpha$

$\max R_{A+B}(\vec{x}) = \max\{A+B$ 的特征值$\} := \beta$

又 $R_A(\vec{x}) \in [\lambda_1, \lambda_n]$

$R_B(\vec{x}) \in [\mu_1, \mu_n]$

$\therefore\ R_{A+B}(\vec{x}) = R_A(\vec{x}) + R_B(\vec{x}) \in [\lambda_1+\mu_1, \lambda_n+\mu_n]$

$\therefore\ [\alpha, \beta] \subseteq [\lambda_1+\mu_1, \lambda_n+\mu_n]$

∴ 特征值都在 $[\lambda_1+\mu_1, \lambda_n+\mu_n]$ □

3. 设 A 是 n 阶斜对称方阵. 证明: A^2 的特征值都是实数且小于等于 0.

解：$\because$ A 是斜对称阵，$\therefore$ $\exists\, P \in O_n(\mathbb{R})$ s.t.

$$P^t A P = \begin{pmatrix} N_2(0,\beta_1) & & & & & \\ & \ddots & & & & \\ & & N_2(0,\beta_s) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} := C$$

其中 $\beta_1, \cdots, \beta_s \in \mathbb{R}$. $N_2(0,\beta_i) = \begin{pmatrix} 0 & -\beta_i \\ \beta_i & 0 \end{pmatrix}$

$$A = P \cdot C P^t \Rightarrow A^2 = P C^2 P^t \qquad A^2 \sim C^2$$

从而 A^2 的特征值与 C^2 的特征值相同.

$$C^2 = \begin{pmatrix} \begin{pmatrix} -\beta_1^2 & 0 \\ 0 & -\beta_1^2 \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{pmatrix} -\beta_s^2 & 0 \\ 0 & -\beta_s^2 \end{pmatrix} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

此时 C^2 的特征值为 $-\beta_1^2, \cdots, -\beta_s^2, 0$.

从而特征值均为实数且 ≤ 0.

法二：A 的特征根为纯虚数或 0.

设 A 的特征根为 $\{\beta_1 i, \beta_2 i, \cdots, \beta_t i, 0\}$

$\because$ $A\vec{x} = \lambda \vec{x}$ $\qquad A^2 \vec{x} = \lambda(\lambda \vec{x}) = \lambda^2 \vec{x}$

$\therefore$ A^2 的特征根为 $\{-\beta_1^2, -\beta_2^2, \cdots, -\beta_t^2, 0\}$

为实数 且 ≤ 0. $\square$

4. 设 V 是一个奇数维内积空间. $\mathcal{A}$ 是 V 上的正交变换, $\det(A) = 1$, 其中 A 是 $\mathcal{A}$ 在 V 的某组基下的矩阵. 证明: $\mathcal{A}$ 有特征值 1.

注: 因为行列式是相似不变量, 所以 $\det(A)$ 的值与基底选取无关.

4. $A \sim_o M = \begin{pmatrix} N(\cos\theta_1, \sin\theta_1) & & & & & \\ & \ddots & & & & \\ & & N(\cos\theta_s, \sin\theta_s) & & & \\ & & & \lambda_{2s+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$

故 $\det(A) = \det(M) = \underbrace{\lambda_{2s+1} \cdots \lambda_n}_{\text{奇数}} = 1$ 其中 $\lambda_{2s+1}, \cdots, \lambda_n \in \{1, -1\}$

若 λ_i 均为 -1 则 $\det(A) = -1$.

$\therefore \exists\, i \in \{2s+1, n\}$ s.t. $\lambda_i = 1$. $\Rightarrow$ M 有特征值 1,

$\Rightarrow$ A 有特征值 1. $\square$

5. 设 $A, B \in \mathrm{O}_n(\mathbb{R})$, $\det(A) + \det(B) = 0$. 证明: $A + B$ 不可逆.

证: $\because A, B \in O_n(\mathbb{R})$ $\therefore \det(A) = \pm 1$, $\det(B) = \pm 1$

又 $\because \det(A) + \det(B) = 0$, $\therefore \det(A)$ 与 $\det(B)$ 异号,

不妨设 $\det(A) = 1$ $\det(B) = -1$

$\det(A+B) = \det(A) \cdot \det(E + A^{-1}B)$

$= \det(E + A^{-1}B)$

$\because O_n(\mathbb{R})$ 是群, $\therefore A^{-1}B \in O_n(\mathbb{R})$. 且 $\det(A^{-1}B) = 1 \cdot (-1) = -1$

由例7.2知 $\det(E+A^{-1}B)=0$ 所以 $\det(A+B)=0$

$\downarrow$

证: $\because A^{-1}B \in O_n(\mathbb{R})$ $\therefore \exists\, Q \in O_n(\mathbb{R})$ s.t.

$$Q^T(A^{-1}B)Q = \begin{pmatrix} N(\cos\theta_1, \sin\theta_1) & & & & & \\ & \ddots & & & & \\ & & N(\cos\theta_s, \sin\theta_s) & & & \\ & & & \lambda_{2s+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix} := M$$

$\because \det(A^{-1}B) = -1$ $\therefore \det(M) = \det(Q)^2 \det(A^{-1}B) = -1$

$\therefore \exists\, j \in \{2s+1, \cdots, n\}$ s.t. $\lambda_j = -1$

$\Rightarrow -1$ 是 M 的特征值, $\Rightarrow -1$ 是 $A^{-1}B$ 的特征值 $\Rightarrow |-1\cdot E - A^{-1}B|$

$= |E + A^{-1}B| = 0$

证二: $\det(A^t(A+B)B^t) = \det(A^t)\det(A+B)\det(B^t)$

$= -\det(A+B)$

$\det(A^t(A+B)B^t) = \det(A^tAB^t + A^tBB^t)$

$= \det(B^t + A^t) = \det((B+A)^t) = \det(B+A)$

$= \det(A+B)$

$\Rightarrow -\det(A+B) = \det(A+B) \Rightarrow \det(A+B) = 0$

$\square$

矩阵同时对角化

定理 5.4：设 $A, B \in SM_n(\mathbb{R})$ 且 A 正定，则存在 $P \in GL_n(\mathbb{R})$ s.t. $P^tAP=E$ 和 P^tBP 是对角阵.

Pf: $\because$ A 正定，$\therefore$ $\exists$ $P_1 \in GL_n(\mathbb{R})$ s.t. $P_1^t A P_1 = E$ （惯性定理）

令 $C = P_1^t B P_1$ （$\because$ 要同时对角化）则 C 对称.

从而 $\exists$ $P_2 \in O_n(\mathbb{R})$ s.t. $D = P_2^t C P_2$ （对称矩阵正交相似于对角阵）

其中 D 是对角阵.

令 $P = P_1 P_2$ 则 $P^t B P = P_2^t C P_2 = D$ 且

$P^t A P = P_2^t E P_2 = P_2^t P = E$ （$\because$ $P_2 \in O_n(\mathbb{R})$）.

例 5.5 设 $A, B \in M_n(\mathbb{R})$，A 正定且 B 半正定，证明：

$$\det(A+B) \geq \det(A) + \det(B)$$

证：由上述定理 $\exists$ $P \in GL_n(\mathbb{R})$ s.t. $P^tAP=E$ 和

$P^tBP = diag(\alpha_1, \cdots, \alpha_n)$ 于是

$P^t(A+B)P = diag(1+\alpha_1, \cdots, 1+\alpha_n)$

两边取行列式 得 $|P|^2 \cdot |A+B| = (1+\alpha_1)\cdots(1+\alpha_n)$

另一方面 $|P|^2 \cdot (|A|+|B|) = |P|^2 \cdot (1+\alpha_1\alpha_2\cdots\alpha_n)$

$\because$ B 半正定 $\therefore$ $\alpha_1, \cdots, \alpha_n$ 非负，于是

$$\prod_{i=1}^n (1+\alpha_i) \geq 1 + \prod_{i=1}^n \alpha_i$$

$$\Rightarrow |A+B| \geq |A|+|B|.$$

$\square$

2. 设 $A, B \in SM_n(\mathbb{R})$, A 正定 则 AB 特征值为实数:

证: A 正定, 断言 A^{-1} 也正定:

证明断言 $\because A$ 正定 $\therefore \exists P \in GL_n(\mathbb{R})$ s.t. $P^tAP = E$

$$\Rightarrow P^{-1}A^{-1}(P^t)^{-1} = E \Rightarrow P^{-1}A^{-1}(P^{-1})^t = E$$

$$\Rightarrow ((P^{-1})^t)^t A^{-1}(P^{-1})^t = E \Rightarrow A^{-1} \text{正定.}$$

所以 $\exists Q \in GL_n(\mathbb{R})$ s.t. $Q^tA^{-1}Q = E$ 且

$Q^tBQ = diag(\alpha_1, \cdots, \alpha_n) := C$ 其中 $C \in M_n(\mathbb{R})$

则 $C = E \cdot C = (Q^tA^{-1}Q)^{-1} \cdot (Q^tBQ)$

$$= Q^{-1}A(Q^t)^{-1}Q^tBQ$$

$$= Q^{-1}ABQ$$

$AB \sim_c C$, $\because C$ 的特征值均为实数. $\therefore AB$ 的特征值均为实数.

例: 设 A 是 n 阶实对称正定矩阵, S 是 n 阶反对称实矩阵, 证:

$$|A+S| > 0$$

解: $\because A \in SM_n(\mathbb{R})$ 且正定, $\therefore \exists P \in GL_n(\mathbb{R})$ s.t. $P^tAP = E_n$,

记 $C = P^tSP$, 则 $P^t(A+S)P = E_n + C$

$$\Rightarrow \det(P)^2 \cdot \det(A+S) = \det(E_n + C)$$

$\because C^t = (P^tSP)^t = P^tS^tP = -P^tSP = -C \Rightarrow C$ 也是反对称矩阵.

断言(例6.2) 设 $C\in SSM_n(\mathbb{R})$. 则 $\det(E+C)\geq 1$

对实反对称矩阵 $\exists\, Q\in O_n(\mathbb{R})$ s.t. $Q^tCQ=M$, $N(0,\beta_1)=\begin{pmatrix}0&-\beta_1\\ \beta_1&0\end{pmatrix}$

$$M=\mathrm{diag}(N(0,\beta_1),\cdots N(0,\beta_s),0,\cdots,0)$$

从而 $Q^t(E+C)Q=\mathrm{diag}(N(1,\beta_1)\cdots N(1,\beta_s),1,\cdots,1)$

$\because$ $\det(Q)=\pm 1$ 且 $\det(N(1,\beta_i))\geq 1$

所以 $\det(E+C)\geq 1$.

又 $\because$ $P\in GL_n(\mathbb{R})$ $\det(P)\neq 0$ $\det(P)^2>0$

$$\Rightarrow \det(A+S)=\frac{1}{\det(P)^2}\det(E+C)>0$$

$\square$