1. 设 $S = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \in \mathrm{SM}_2(\mathbb{R})$.

(1) 是否存在 $P \in \mathrm{GL}_2(\mathbb{R})$ 使得 $S = P^tP$? 如果存在, 计算这样的矩阵 P;

(2) 是否存在 $P \in \mathrm{GL}_2(\mathbb{C})$ 使得 $S = P^tP$? 如果存在, 计算这样的矩阵 P.

解: (1)

$$(S \mid E) = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{array}\right) \xrightarrow{(2)+(1)} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1\\ 1 & 0 & 0 & 1\end{array}\right) \xrightarrow{\text{对称操作}} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 1\\ 1 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}(1)+(2)} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 1\\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right) \xrightarrow{\text{对称操作}} \left(\begin{array}{cc|cc} 2 & 0 & 1 & 1\\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right)$$

S 的签名为 $(1,1)$ 不存在 $p \in GL_2(\mathbb{R})$ s.t. $S = p^T \cdot p$.

(2)

$$\left(\begin{array}{cc|cc} 2 & 0 & 1 & 1\\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right) \xrightarrow{\frac{1}{\sqrt{2}}(1)} \left(\begin{array}{cc|cc} \sqrt{2} & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right) \xrightarrow{\text{对称操作}} \left(\begin{array}{cc|cc} 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}\right)$$

$$\xrightarrow{\sqrt{2}i(2)} \left(\begin{array}{cc|cc} 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ 0 & -\frac{1}{\sqrt{2}}i & -\frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2}i\end{array}\right) \xrightarrow{\text{对称操作}} \left(\begin{array}{cc|cc} 1 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ 0 & 1 & -\frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2}i\end{array}\right)$$

令 $p = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}i\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i\end{pmatrix}$ $\quad p^TSp = I \Rightarrow S = (p^{-1})^t \cdot p^{-1}$.

$$P^{-1} = \frac{1}{i}\begin{pmatrix} \frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2}i \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i & -\frac{\sqrt{2}}{2}i \end{pmatrix}$$

分析可得：$|S| = -1$ 故 $\nexists P \in Gl_n(\mathbb{R})$ s.t. $S = P^T \cdot P$.

$\exists P \in Gl_n(\mathbb{C})$ s.t. $S = P^T \cdot P$.

也可以用待定系数法...

2. 设 $A \in \mathrm{SM}_n(\mathbb{R})$ 且 $\det(A) < 0$. 证明：存在 $\boldsymbol{x} \in \mathbb{R}^n$ 使得 $\boldsymbol{x}^t A \boldsymbol{x} < 0$.

证明：设 A 的规范型为 $\begin{pmatrix} E_s & & \\ & -E_t & \\ & & 0 \end{pmatrix}$ $\exists P \in Gl_n(\mathbb{R})$ s.t. $P^t A P = \begin{pmatrix} E_s & & \\ & -E_t & \\ & & 0 \end{pmatrix} =: B$

等式左边取行列式可得：$|P|^2 \cdot |A| < 0$. 故 $t > 0$

取 $\vec{y} = (0, \cdots, 0, \underset{\substack{\uparrow \\ \text{第}s+1\text{个分量}}}{1}, 0, \cdots)^t$. 那么 $\vec{y}^t B \vec{y} = -1 < 0$.

$\vec{y}^t \cdot B \cdot \vec{y} = \vec{y}^t P^t A P \cdot \vec{y} = \vec{x}^t \cdot A \cdot \vec{x} < 0$. 其中 $\vec{x} = P \cdot \vec{y}$. $\square$.

反证法：假设 $\forall \vec{x} \in \mathbb{R}^n$，都有 $\vec{x}^t A \vec{x} \geqslant 0$，即 A 为半正定矩阵.

结合上周作业，半正定矩阵的行列式非负 $|A| \geqslant 0$. $\rightarrow\leftarrow$.

3. 设 $A \in \mathrm{SM}_n(\mathbb{R})$ 正定. 证明：对于任意的 $m \in \mathbb{Z}$，A^m 正定.

证明：数学归纳法，当 $m=1$ 时结论显然成立.

设当 $m=n \in \mathbb{N}^*$ 时结论成立，考虑当 $m=n+1$ 时

由 A 正定 $\exists\, P \in GL_n(\mathbb{R})$ s.t. $A = P^t \cdot P$

$$A^{n+1} = A^n \cdot A = P^t \cdot \underbrace{P \cdot P^t \cdot P \cdots P^t \cdot P \cdot P^t}\cdot P$$

$$A^{n+1} \sim_c (P^t \cdot P)^n. \quad \text{令 } B = P \cdot P^t.$$

$$\overset{B}{P \cdot P^T} \sim_c A^2 \sim_c \overset{B^3}{(P \cdot P^T)^3} \sim_c A^4$$

$$A \sim_c \underset{B^2}{(P \cdot P^T)^2} \sim_c A^3 \sim_c \underset{B^4}{(P \cdot P^T)^4}$$

$\because$ B 正定 $\therefore$ 由归纳假设 B^n 正定，故 A^{n+1} 正定.

当 $m<0$ 时 $A^m = (A^{-m})^{-1}$

断言：若 $C \in \mathrm{SM}_n(\mathbb{R})$ 正定，则 C^{-1} 正定.

$\because$ $C \in \mathrm{SM}_n(\mathbb{R})$ 正定 $\therefore$ $\exists\, Q \in GL_n(\mathbb{R})$ s.t. $C = Q^t \cdot Q$.

由此 $C^{-1} = Q^{-1}(Q^t)^{-1} = Q^{-1}(Q^{-1})^t$ 故 C^{-1} 也是正定矩阵.

另：由 $A \in SM_n(\mathbb{R})$ 正定 得 $A^t = A$、$|A| \neq 0$.

$\forall m \in \mathbb{Z}^+$、假设 A^n 正定对任意小于 m 的正整数 n 都成立

若 $m = 2k$，则 $A^m = A^k \cdot A^k = (A^k)^t \cdot A^k$. $A^m \sim_c I_n$. A^m 是正定矩阵. A 对称，$A \in GL_n(\mathbb{R})$.

若 $m = 2k+1$，则 $A^m = A^{2k+1} = A^k \cdot A \cdot A^k \Rightarrow A^m \sim_c A$

$\forall m \in \mathbb{Z}_{<0}$ A^{-m} 正定 $\Rightarrow (A^{-m})^{-1} = A^m$ 正定.

等价于下面写法：

$\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$, $\vec{x}^t A^{2k} \vec{x} = (A^k \cdot \vec{x})^t \cdot (A^k \vec{x}) > 0$.

$\vec{x}^t \cdot A^{2k+1} \cdot \vec{x} = (A^k \vec{x})^t \cdot A \cdot (A^k \vec{x}) > 0$

4. 求实函数 $p(\mathbf{x}) = \mathbf{x}^t A\mathbf{x} + \boldsymbol{\alpha}\mathbf{x} + a$ 的最小值, 其中 A 是 n 阶正定实对称方阵, $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ 是 n 维实行向量, a 是实数.

解：由 A 正定 则 $\exists P \in GL_n(\mathbb{R})$ s.t. $A = P^t \cdot P$. 令 $\vec{y} = P \cdot \vec{x}$，$\vec{\beta} = \vec{\alpha} \cdot P^{-1} = (b_1, \cdots, b_n)$ 则

可逆的坐标变换.

$$
\begin{aligned}
p(\vec{x}) &= \vec{x}^t A \vec{x} + \vec{\alpha} \vec{x} + a \\
&= \vec{x}^t \cdot P^t \cdot P \cdot \vec{x} + \vec{\alpha} \cdot \vec{x} + a \\
&= \vec{y}^t \cdot \vec{y} + \vec{\alpha} \cdot P^{-1} \cdot \vec{y} + a.
\end{aligned}
$$

$$= \vec{y}^t \cdot \vec{y} + \vec{\beta} \cdot \vec{y} + a.$$

$$= y_1^2 + \cdots + y_n^2 + b_1 y_1 + \cdots + b_n y_n + a.$$

$$= (y_1 + \frac{b_1}{2})^2 + \cdots + (y_n + \frac{b_n}{2})^2 + a - \sum_{i=1}^{n} \frac{1}{4} b_i^2$$

故当 $\vec{y} = -\frac{1}{2}\vec{\beta}^t = -\frac{1}{2}(P^{-1})^t \vec{\alpha}^t$ 时，即 $\vec{x} = -\frac{1}{2} P^{-1}(P^{-1})^t \cdot \vec{\alpha}^t = -\frac{1}{2} \cdot A^{-1} \cdot \vec{\alpha}^t$

$p(\vec{x})$ 取最小值 $a - \sum_{i=1}^{n} \frac{1}{4} b_i^2 = a - \frac{1}{4} \vec{\alpha} \cdot P^{-1}(P^{-1})^t \vec{\alpha}^t = a - \frac{1}{4} \vec{\alpha} \cdot A^{-1} \cdot \vec{\alpha}^t.$

矩阵解法: $p(\vec{x}) = (\vec{x}^t, 1) \begin{pmatrix} A & \frac{1}{2}\vec{\alpha}^t \\ \frac{1}{2}\vec{\alpha} & a \end{pmatrix} \begin{pmatrix} \vec{x} \\ 1 \end{pmatrix}$，令 $B = \begin{pmatrix} A & \frac{1}{2}\vec{\alpha}^t \\ \frac{1}{2}\vec{\alpha} & a \end{pmatrix}$

$|A| \neq 0$

$$\underbrace{\begin{pmatrix} E_n & 0 \\ -\frac{1}{2}\vec{\alpha} A^{-1} & 1 \end{pmatrix}}_{P^t} B \underbrace{\begin{pmatrix} E_n & -\frac{1}{2} A^{-1} \vec{\alpha}^t \\ \vec{0} & 1 \end{pmatrix}}_{P} = \begin{pmatrix} A & 0 \\ 0 & a - \frac{1}{4}\vec{\alpha} A^{-1} \vec{\alpha}^t \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} E_n & \frac{1}{2} A^{-1} \vec{\alpha}^t \\ \vec{0} & 1 \end{pmatrix}$$

$$p(\vec{x}) = (\vec{x}^t, 1) \cdot (P^t)^{-1} \begin{pmatrix} A & 0 \\ 0 & a - \frac{1}{4}\vec{\alpha} A^{-1} \vec{\alpha}^t \end{pmatrix} P^{-1} \begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} = (\vec{x}^t + \frac{1}{2}\vec{\alpha} A^{-1}, 1) \cdot \begin{pmatrix} A & 0 \\ 0 & a - \frac{1}{4}\vec{\alpha} A^{-1} \vec{\alpha}^t \end{pmatrix} \cdot \begin{pmatrix} \vec{x} + \frac{1}{2} A^{-1} \vec{\alpha}^t \\ 1 \end{pmatrix}$$

由 A 正定，当 $\vec{x} + \frac{1}{2} A^{-1} \vec{\alpha}^t = \vec{0}$ 即 $\vec{x} = -\frac{1}{2} A^{-1} \vec{\alpha}^t$ 时，$p(\vec{x})$ 取最小值 $a - \frac{1}{4}\vec{\alpha} A^{-1} \vec{\alpha}^t$.

微积分: 若 $\nabla p(\vec{x}) = \left(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_n}\right)\Big|_{\vec{x} = \vec{x}_0} = \vec{0}$，则 $\vec{x}_0$ 可能是极值点，需进一步分析，Hessian 矩阵……极大、小、鞍点.

5. 设 $B \in \mathrm{SM}_{n-1}(\mathbb{R})$ 正定. $\boldsymbol{v} \in \mathbb{R}^{n-1}$ 且 $a \in \mathbb{R}$, 令

$$A = \begin{pmatrix} B & \boldsymbol{v} \\ \boldsymbol{v}^t & a \end{pmatrix}.$$

证明: 如果 $\det(A) = 0$, 则 A 半正定.

证明: 由 $B \in SM_{n-1}(\mathbb{R})$ 正定, 存在 $P \in GL_n(\mathbb{R})$ s.t. $P^t B P = I$.

$$\begin{pmatrix} P^t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & \vec{v} \\ \vec{v}^t & a \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P^t B P & P^T \vec{v} \\ \vec{v}^t P & a \end{pmatrix} = \begin{pmatrix} E_{n-1} & P^t \vec{v} \\ \vec{v}^t P & a \end{pmatrix}$$

$$\begin{pmatrix} E_{n-1} & 0 \\ -\vec{v}^t P & 1 \end{pmatrix}\begin{pmatrix} E_{n-1} & P^t \vec{v} \\ \vec{v}^t P & a \end{pmatrix}\begin{pmatrix} E_{n-1} & -P^t \vec{v} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} E_{n-1} & 0 \\ 0 & a - \vec{v}^t P \cdot P^t \vec{v} \end{pmatrix}$$

$|P|^2 \cdot |A| = a - v^t P \cdot P^t \vec{v} = 0$ $\because |P|^2 \neq 0$. $\therefore a - v^t P \cdot P^t \vec{v} = 0$

故 $A \sim_c \begin{pmatrix} E_{n-1} & \\ & 0 \end{pmatrix}$ 由此 A 半正定.

注: 应用Jacobi公式注意前提条件. $A \in SM_n(F)$ $\Delta_0 = 1$ 若 $\Delta_i \neq 0$ $\forall i = 1, \cdots, n$ 则

$A \simeq_c \mathrm{diag}\left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \cdots, \frac{\Delta_n}{\Delta_{n-1}}\right)$

另：

Laplace展开.

设 $A(\lambda) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}+\lambda \end{pmatrix}$ $\quad |A(\lambda)| = \begin{vmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda \end{vmatrix}$

$= |A| + |A_{n-1}| \cdot \lambda$

$= 0 + \underset{>0}{|A_{n-1}|}\lambda.$

当 $\lambda > 0$ 时 $|A(\lambda)| > 0$. 由 $\Delta_i(A(\lambda)) = \Delta_i(A) > 0 \ \forall i = 1, \cdots, n-1$. 得 $A(\lambda)$ 正定.

当 $\lambda > 0$ 时 $\forall X \neq \vec{0}$ 有 $\vec{x}^t A(\lambda) \vec{x} > 0$. 令 λ 在区间 $(0, +\infty)$ 内趋于 0 得 $\vec{x}^t A \vec{x} \geq 0$. 得到 A 半正定.

另：$A = \begin{pmatrix} B & \vec{v} \\ \vec{v}^t & a \end{pmatrix}$ $\quad \begin{pmatrix} E_{n-1} & 0 \\ -\vec{v}^t B^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} B & \vec{v} \\ \vec{v}^t & a \end{pmatrix} = \begin{pmatrix} B & \vec{v} \\ 0 & a - \vec{v}^t B^{-1} \vec{v} \end{pmatrix}$

$\det(A) = \det(B) \cdot (a - \vec{v}^t B^{-1} \vec{v}) = 0. \Rightarrow a = \vec{v}^t B^{-1} \vec{v}.$

$\forall \vec{x} \in \mathbb{R}^n \quad \vec{x} = \begin{pmatrix} \vec{y} \\ z \end{pmatrix} \quad \vec{x}^t A \vec{x} = (\vec{y}^t \ z) \begin{pmatrix} B & \vec{v} \\ \vec{v}^t & a \end{pmatrix} \begin{pmatrix} \vec{y} \\ z \end{pmatrix}$

$= (\vec{y}^t B + z\vec{v}^t \quad \vec{y}^t \vec{v} + az) \begin{pmatrix} \vec{y} \\ z \end{pmatrix}$

$= \vec{y}^t B \vec{y} + z\vec{v}^t \vec{y} + z\vec{y}^t \vec{v} + az^2$

$= (\vec{y}^t + z\vec{v}^t B^{-1}) \cdot B (\vec{y} + z B^{-1} \vec{v}) \geqslant 0.$

复习：线性映射下的矩阵：

$\phi \in \mathrm{Hom}(V, W)$. V, W 是F上的线性空间. $\vec{e}_1, \cdots, \vec{e}_n$ 是 V 的一组基. $\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m$ 是 W 的一组基.

$\phi(\vec{e}_j) = a_{1,j} \vec{\varepsilon}_1 + a_{2,j} \vec{\varepsilon}_2 + \cdots + a_{m,j} \vec{\varepsilon}_m = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) \underbrace{\begin{pmatrix} a_{1,j} \\ \vdots \\ a_{m,j} \end{pmatrix}}_{\vec{A}_j}$

定义：

$(\phi(\vec{e}_1), \cdots, \phi(\vec{e}_n)) = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) \cdot A \quad A \in F^{m \times n}$

$\vec{x} = (\vec{e}_1 \cdots \vec{e}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{y} = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

$$\phi(\vec{x})=\vec{y} \Leftrightarrow (\phi(\vec{\varepsilon}_1),\cdots,\phi(\vec{\varepsilon}_n))\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=(\vec{\varepsilon}_1,\cdots\vec{\varepsilon}_m)\begin{pmatrix}y_1\\ \vdots\\ y_m\end{pmatrix}$$

$$\Leftrightarrow (\vec{\varepsilon}_1,\cdots,\vec{\varepsilon}_m)\cdot A\cdot\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=(\vec{\varepsilon}_1\cdots\vec{\varepsilon}_m)\begin{pmatrix}y_1\\ \vdots\\ y_m\end{pmatrix}$$

$$\Leftrightarrow A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}y_1\\ \vdots\\ y_m\end{pmatrix}$$ 坐标表示.

设 $A\in F^{m\times n}$. $\phi: F^{n\times 1}\to F^{m\times 1}$ $\forall X\in F^{n\times 1}$, $\phi(X)=AX$. ϕ在标准基下的矩阵为A.

例1.6. 设$A\in F^{m\times n}$. $\phi: F^{n\times k}\to F^{m\times k}$ 由公式$\forall X\in F^{n\times k}$, $\phi(X)=AX$ 给出.求ϕ在标准基下的矩阵.

$\forall X\in F^{n\times k}$ X在标准基 $E_{1,1},\cdots,E_{n,1},\cdots,E_{1,k},\cdots,E_{n,k}$.下的坐标为.

$\begin{pmatrix}\vec{X}_1\\ \vec{X}_2\\ \vdots\\ \vec{X}_k\end{pmatrix}_{nk\times 1}$, $\phi(X)=AX$ 在标准基$E_{1,1},\cdots,E_{m,1},\cdots,E_{1,k},\cdots,E_{m,k}$下的坐标为, $\begin{pmatrix}\vec{AX}_1\\ \vdots\\ \vec{AX}_k\end{pmatrix}=\begin{pmatrix}A\vec{X}_1\\ \vdots\\ A\vec{X}_k\end{pmatrix}_{mk\times 1}$

$$AX=A(\vec{X}_1,\cdots,\vec{X}_k)=(A\vec{X}_1,\cdots,A\vec{X}_k)$$

$$\begin{pmatrix} A\vec{X}_1 \\ \vdots \\ A\vec{X}_k \end{pmatrix}_{mk\times 1} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}_{mk\times nk} \cdot \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vdots \\ \vec{X}_k \end{pmatrix}_{nk\times 1}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F^{2\times 2} \quad \phi: M_2(F) \to M_2(F)$$

$$X \mapsto AX$$

$$X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

求ϕ在标准基 $E_{1,1}$、$E_{2,1}$、$E_{1,2}$, $E_{2,2}$ 下的矩阵:

$$\phi(E_{1,1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

$$\phi(E_{2,1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}$$

$$\phi(E_{1,2}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$$

$$\phi(E_{2,2}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

$$\Phi = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

复习：线性映射在不同基底下的矩阵.

$\phi \in Hom(V, W)$. 设 $\vec{e}_1', \cdots, \vec{e}_n'$ 是 V 的另一组基，$\vec{\varepsilon}_1', \cdots, \vec{\varepsilon}_m'$ 是 W 的另一组基.

$(\vec{e}_1', \cdots, \vec{e}_n') = (\vec{e}_1, \cdots, \vec{e}_n) \cdot P$ 和 $(\vec{\varepsilon}_1', \cdots, \vec{\varepsilon}_m') = (\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m) \cdot Q$.

其中 $P \in GL_n(F)$，$Q \in GL_m(F)$. 如果 ϕ 在 $\vec{e}_1, \cdots, \vec{e}_n$，$\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_m$ 下的矩阵为 A. 则 ϕ 在 $\vec{e}_1', \cdots, \vec{e}_n'$，$\vec{\varepsilon}_1', \cdots, \vec{\varepsilon}_m'$ 下的矩阵是 $Q^{-1}AP$.

$$\vec{x} = (\vec{e}_1', \cdots, \vec{e}_n') \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} \qquad \phi(\vec{x}) = \vec{y}, \quad \vec{y} = (\vec{\varepsilon}_1', \cdots, \vec{\varepsilon}_m') \begin{pmatrix} y_1' \\ \vdots \\ y_m' \end{pmatrix}$$

$$= (\vec{e}_1, \cdots, \vec{e}_n) \cdot P \cdot \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} \qquad \phi(\vec{x}) = (\vec{\varepsilon}_1 \cdots \vec{\varepsilon}_m) \cdot A \cdot P \cdot \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = (\vec{\varepsilon}_1', \cdots, \vec{\varepsilon}_m') \, Q^{-1} A \, P \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

打洞引理: $A \sim_e \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \exists Q, P \in GL_n(F)$ s.t. $QAP = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$

取 $(\vec{e}_1', \dots, \vec{e}_n') = (\vec{e}_1, \dots, \vec{e}_n) \cdot P$. $(\vec{\varepsilon}_1', \dots, \vec{\varepsilon}_m') = (\vec{\varepsilon}_1, \dots, \vec{\varepsilon}_m) \cdot Q^{-1}$

则 φ 在 V 的一组基 $\{\vec{e}_1', \dots, \vec{e}_n'\}$ W 的一组基 $\{\vec{\varepsilon}_1', \dots, \vec{\varepsilon}_m'\}$ 使得 φ 在这两组基下的矩阵表示为 $\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$

取 $\ker(\varphi)$ 的一组基 $\{\vec{e}_{r+1}', \dots, \vec{e}_n'\}$ 扩充为 V 的一组基 $\{\vec{e}_1', \dots, \vec{e}_n'\}$.

取 W 的一组基 $\{\underbrace{\varphi(\vec{e}_1')}_{=\vec{\varepsilon}_1'}, \dots, \underbrace{\varphi(\vec{e}_r')}_{=\vec{\varepsilon}_r'}, \vec{\varepsilon}_{r+1}', \dots, \vec{\varepsilon}_m'\}$.

Jacobi 公式应用举例:

求二次型 $f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + 4 \sum_i \sum_j x_i x_j$ 的秩与签名.

$$|A_n| = \begin{vmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 \end{vmatrix} \xrightarrow[j=2,\cdots,n]{(j)+(1)} \begin{vmatrix} 1+2(n-1) & 2 & \cdots & 2 \\ 1+2(n-1) & 1 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 1+2(n-1) & 2 & \cdots & 1 \end{vmatrix} = (2n-1) \begin{vmatrix} 1 & 2 & \cdots & 2 \\ 1 & 1 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 1 & 2 & \cdots & 1 \end{vmatrix}$$

$$\xrightarrow[-(1)+(i)]{i=2,\cdots,n} (2n-1) \begin{vmatrix} 1 & 2 & \cdots & 2 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix} = (2n-1)(-1)^{n-1} \neq 0$$

令 $\Delta_0 := 1$, $\Delta_1 = 1$, $\frac{\Delta_{i+1}}{\Delta_i} = -\frac{2i+1}{2i-1} < 0$ $\quad i = 1, \dots, n-1$ $\quad A \sim_c \operatorname{diag}\left(\frac{\Delta_1}{\Delta_0}, \cdots, \frac{\Delta_n}{\Delta_{n-1}}\right)$ $\quad A$ 的签名为 $(1, n-1)$.