第十一次作业

1. 计算下列行列式的值:

$$\det\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix} \quad \text{和} \quad \det\begin{pmatrix}0 & 1 & 1 & 8\\ 3 & 1 & -1 & 9\\ 0 & -1 & 1 & 0\\ 0 & -1 & -1 & 0\end{pmatrix}.$$

2. 展开下列行列式:

$$D_n = \det\begin{pmatrix}1 & 2 & 3 & \cdots & n-1 & n\\ 2 & 3 & 4 & \cdots & n & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ n-1 & n & 1 & \cdots & n-3 & n-2\\ n & 1 & 2 & \cdots & n-2 & n-1\end{pmatrix} \quad \text{和} \quad \Delta_n = \det\begin{pmatrix}2a & 1 & 0 & \cdots & 0 & 0\\ a^2 & 2a & 1 & \cdots & 0 & 0\\ 0 & a^2 & 2a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2a & 1\\ 0 & 0 & 0 & \cdots & a^2 & 2a\end{pmatrix}.$$

3. 证明:

$$D_{2n} = \det\begin{pmatrix}a_n & & & & & & & b_n\\ & a_{n-1} & & & & & b_{n-1} & \\ & & \ddots & & & \iddots & & \\ & & & a_1 & b_1 & & & \\ & & & c_1 & d_1 & & & \\ & & \iddots & & & \ddots & & \\ & c_{n-1} & & & & & d_{n-1} & \\ c_n & & & & & & & d_n\end{pmatrix} = \prod_{i=1}^{n}\det\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}.$$

4. 设 $A, B, C \in \mathrm{M}_n(\mathbb{R})$. 利用矩阵分块法证明:

$$\mathrm{rank}(AB) + \mathrm{rank}(BC) - \mathrm{rank}(B) \leq \mathrm{rank}(ABC).$$

5. 设 $A = (a_{i,j}) \in \mathrm{M}_n(\mathbb{R})$ 且对任意 $i \in \{1, 2, \ldots, n\}$, 我们有

$$|a_{i,i}| > \sum_{j=1, j\neq i}^{n} |a_{i,j}|.$$

证明: $\det(A) \neq 0$.