

第二次习题课

$$1. \begin{cases} a\alpha_1^3 + b\alpha_1^2 + c\alpha_1 + d = \beta_1 \\ a\alpha_2^3 + b\alpha_2^2 + c\alpha_2 + d = \beta_2 \\ a\alpha_3^3 + b\alpha_3^2 + c\alpha_3 + d = \beta_3 \\ a\alpha_4^3 + b\alpha_4^2 + c\alpha_4 + d = \beta_4 \end{cases}$$

系数矩阵

$$A = \begin{pmatrix} \alpha_1^3 & \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^3 & \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^3 & \alpha_3^2 & \alpha_3 & 1 \\ \alpha_4^3 & \alpha_4^2 & \alpha_4 & 1 \end{pmatrix}$$

注意: a, b, c, d 是变元

增广矩阵

$$M = \left(A \begin{array}{c} \vdots \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_4 \end{array} \right)$$

2. (1) 求证 $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2$

① $n=1$ 时, $\sum_{k=1}^1 k^3 = 1 = \left(\frac{1 \times 2}{2} \right)^2$, 成立

② 假设 $n-1$ ($n > 1$) 时成立, 则

$$\sum_{k=1}^n k^3 = n^3 + \sum_{k=1}^{n-1} k^3 = n^3 + \left(\frac{(n-1)n}{2} \right)^2$$

$$= n^3 + \frac{n^2(n-1)^2}{4}$$

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left(\frac{n(n+1)}{2} \right)^2 \quad \square$$

$$(2) \text{ 求证 } (x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

$$\textcircled{1} \quad n=1 \text{ 时, } (x+y)^1 = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x+y$$

$\textcircled{2}$ 假设当 $n < k$ 时成立, 则当 $n = k$ 时,

$$(x+y)^k = (x+y)(x+y)^{k-1}$$

$$= (x+y) \left(\sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-1-i} y^i \right)$$

$$= \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-i} y^i + \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-1-i} y^{i+1}$$

$$= \binom{k-1}{0} x^k + \binom{k-1}{1} x^{k-1} y + \dots + \binom{k-1}{k-1} x y^{k-1}$$

$$+ \binom{k-1}{0} x^{k-1} y + \dots + \binom{k-1}{k-2} x y^{k-1} + \binom{k-1}{k-1} y^k$$

$$= x^k + \binom{k}{1} x^{k-1} y + \dots + \binom{k}{k-1} x y^{k-1} + y^k$$

$$= \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i$$

组合数公式

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1} \quad (n \geq m+1)$$

□

$$3. (1) \begin{cases} x_2 + x_3 + x_4 = 1, \\ x_1 + x_3 + x_4 = 2, \\ x_1 + x_2 + x_4 = 3, \\ x_1 + x_2 + x_3 = 4 \end{cases}$$

增广矩阵为:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 4 \end{pmatrix}$$

对换(1).(2)行

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 4 \end{pmatrix}$$

(3)-(1), (4)-(1)

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 2 \end{pmatrix}$$

(3)-(2), (4)-(2)

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{pmatrix}$$

(4) - $\frac{1}{2}$ (3)

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 1 \end{pmatrix}$$

相容且确定

判断线性方程组的相容和确定时, 是同时考虑系数矩阵和增广矩阵.
即是对增广矩阵做初等行变换.

$$(2) \begin{cases} -2x_1 + x_2 + 2x_3 = 1, \\ 4x_1 + 7x_2 + 4x_3 = 1, \\ 5x_1 + 2x_2 - x_3 = -1 \end{cases}$$

增广矩阵为:

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 4 & 7 & 4 & 1 \\ 5 & 2 & -1 & -1 \end{array} \right) \xrightarrow{(2) + 2(1), (3) + \frac{5}{2}(1)} \left(\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 0 & 9 & 8 & 3 \\ 0 & \frac{9}{2} & 4 & \frac{3}{2} \end{array} \right)$$

$$\xrightarrow{(2) - 2(3)} \left(\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{9}{2} & 4 & \frac{3}{2} \end{array} \right) \xrightarrow{2 \times (3)} \left(\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & 8 & 3 \end{array} \right)$$

对换(2),(3) $\rightarrow$
$$\begin{pmatrix} -2 & 1 & 2 & 1 \\ 0 & 9 & 8 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

相容但不确定

4. 设该齐次线性方程组为

$$H: \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

系数矩阵为

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$\therefore x_1 = \alpha_1, x_2 = \alpha_2, \dots, x_n = \alpha_n$ 和 $x_1 = \beta_1, x_2 = \beta_2, \dots, x_n = \beta_n$ 是 L 的
两组解

$$\therefore A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0, \quad A \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = 0.$$

$$\text{则} \quad A \begin{pmatrix} u\alpha_1 + v\beta_1 \\ u\alpha_2 + v\beta_2 \\ \vdots \\ u\alpha_n + v\beta_n \end{pmatrix} = A \begin{pmatrix} u\alpha_1 \\ u\alpha_2 \\ \vdots \\ u\alpha_n \end{pmatrix} + A \begin{pmatrix} v\beta_1 \\ v\beta_2 \\ \vdots \\ v\beta_n \end{pmatrix} = u A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + v A \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = 0$$

$\therefore x_1 = u\alpha_1 + v\beta_1, x_2 = u\alpha_2 + v\beta_2, \dots, x_n = u\alpha_n + v\beta_n$ 也是 H 的
解 □

结论：对于齐次线性方程组，解的线性组合、解的倍数亦为解。

几何解释：过原点的平面之交，若不只有原点，则这是过原点直线 or

过原点平面. 端点在这样直线或平面上的向量显然具备上述性质.

4⁺: 若 H 是非齐次的, 证: 对 $\forall k \in \mathbb{R}$,

$$x_1 = \alpha_1 + k(\alpha_1 - \beta_1), \dots, x_n = \alpha_n + k(\alpha_n - \beta_n)$$

也是 H 的解.

$$\begin{aligned} \text{证: } A \begin{pmatrix} \alpha_1 + k(\alpha_1 - \beta_1) \\ \vdots \\ \alpha_n + k(\alpha_n - \beta_n) \end{pmatrix} &= A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + k A \begin{pmatrix} \alpha_1 - \beta_1 \\ \vdots \\ \alpha_n - \beta_n \end{pmatrix} \\ &= \vec{b} + k \left(A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} - A \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \right) \\ &= \vec{b} \end{aligned}$$

数学归纳法: 以自然数的皮亚诺公理系统为基础.

取 $0 \in \mathbb{N}$, 定义后继映射:

$$S: \mathbb{N} \rightarrow \mathbb{N}.$$

$S(n)$ 表示 n 的下一个自然数. 如: $S(0)=1, S(1)=2, \dots$

一组常见的皮亚诺公理可写为:

1. 0 是自然数:

$$0 \in \mathbb{N}.$$

2. 每个自然数都有唯一后继:

$$n \in \mathbb{N} \Rightarrow S(n) \in \mathbb{N}.$$

3. 0 不是任何自然数的后继:

$$\forall n \in \mathbb{N}, S(n) \neq 0.$$

4. 后继映射是单射:

$$S(m) \neq S(n) \Rightarrow m \neq n.$$

5. 归纳公理: 若集合 $A \subseteq \mathbb{N}$ 满足

$$0 \in A$$

且

$$n \in A \Rightarrow S(n) \in A,$$

则

$$A = \mathbb{N}.$$

$\Rightarrow$ 数学归纳法本质

从归纳公理到数学归纳法

假如想证明一个命题 $P(n)$ 对所有自然数 n 成立, 则可定义集合

$$A = \{n \in \mathbb{N} : P(n) \text{ 成立} \}$$

若我们能证明:

$$P(0)$$

以及

$$P(n) \Rightarrow P(n+1),$$

那么就能等价于证明:

$$0 \in A$$

且

$$n \in A \Rightarrow S(n) \in A$$

根据皮亚诺归纳公理:

$$A = \mathbb{N}$$

故数学归纳法

$$P(0), \quad P(n) \Rightarrow P(n+1)$$

来源于自然数系统的公理结构.

例: $\sqrt{6} n_2$:

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

设 $P(n)$: $1+2+\dots+n = \frac{n(n+1)}{2}$

$n=1$ 时:

$$1 = \frac{1 \times 2}{2} = 1$$

假设 $P(n)$ 成立, 即

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

则

$$\begin{aligned} 1+2+\dots+n+n+1 &= \frac{n(n+1)}{2} + n+1 = (n+1) \left(\frac{n}{2} + 1 \right) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

所以:

$$P(n) \Rightarrow P(n+1)$$

由数学归纳法

$P(n)$

对所有 $n \geq 1$ 成立

□

例: 令 $S(n) = \sin(\varphi) + \sin(2\varphi) + \dots + \sin(n\varphi)$, 求证

$$S(n) = \frac{\sin\left(\frac{n\alpha}{2}\right) \sin\left(\frac{(n+1)}{2}\varphi\right)}{\sin\left(\frac{\varphi}{2}\right)} \quad (1)$$

证明: 回顾三角函数和差公式

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \Rightarrow \sin(2a) = 2\sin(a)\cos(a)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \Rightarrow \cos(2a) = \cos^2(a) - \sin^2(a)$$

① $n=1$ 时, $S(1) = \sin(\alpha) = \frac{\sin\left(\frac{\alpha}{2}\right) \sin(\alpha)}{\sin\left(\frac{\alpha}{2}\right)}$, (1) 式成立.

② 假设 $n=k$ 时, (1) 式成立, 即

$$S(k) = \frac{\sin\left(\frac{k\alpha}{2}\right) \sin\left(\frac{(k+1)}{2}\alpha\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

则

$$\begin{aligned} S(k+1) &= S(k) + \sin((k+1)\alpha) \\ &= \frac{\sin\left(\frac{k\alpha}{2}\right) \sin\left(\frac{(k+1)}{2}\alpha\right)}{\sin\left(\frac{\alpha}{2}\right)} + \sin((k+1)\alpha) \end{aligned}$$

$$= \frac{\sin\left(\frac{k\alpha}{2}\right) \sin\left(\frac{(k+1)\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) \sin((k+1)\alpha)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\sin\left(\frac{k\alpha}{2}\right) = \sin\left(\frac{(k+1)\alpha}{2} - \frac{\alpha}{2}\right)$$

$$= \sin\left(\frac{(k+1)\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{(k+1)\alpha}{2}\right) \sin\frac{\alpha}{2}$$

$$= \frac{\sin\left(\frac{k\alpha}{2}\right) \sin\left(\frac{(k+1)\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) \times 2 \times \sin\left(\frac{(k+1)\alpha}{2}\right) \cos\left(\frac{(k+1)\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{(k+1)\alpha}{2}\right) \left[\left(\sin\left(\frac{(k+1)\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{(k+1)\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) \right) + 2 \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{(k+1)\alpha}{2}\right) \right]}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{(k+1)\alpha}{2}\right) \left[\sin\left(\frac{(k+1)\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{(k+1)\alpha}{2}\right) \right]}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{(k+1)\alpha}{2}\right) \sin\left(\frac{(k+2)\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

即 $n = k+1$ 时成立. 由数学归纳法, 对 $\forall n \in \mathbb{N}^*$, (1) 式成立. $\square$