



Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Complete reduction for derivatives in a transcendental Liouvillian extension [☆]

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ARTICLE INFO

Article history:

Received 1 February 2026

Received in revised form 2 September 2026

Accepted 6 October 2026

Available online 9 October 2026

Dataset link: <https://risc.jku.at/sw/reductionpair/>

Keywords:

Complete reduction
Elementary integration
In-field integration
Liouvillian extension
Risch equation
Symbolic integration

ABSTRACT

Transcendental Liouvillian extensions are differential fields in which one can model polylogarithmic, hyperexponential, and trigonometric functions, logarithmic integrals, and their (nested) rational expressions. For such an extension, we construct a complement, over the subfield of constants, to the subspace of derivatives and develop an algorithm that decomposes any element of the field into the sum of a derivative and a component in this complement. Consequently, an element is a derivative if and only if its complementary component vanishes. Moreover, the algorithm enables us to determine elementary integrability over the extension by computing parametric logarithmic parts. It also yields a reduction-based approach to constructing telescopers for elements of the extension, provided that an a priori order bound is available.

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[☆] This research was funded in part by the National Key R&D Program of China (No. 2023YFA1009401), the National Natural Science Foundation of China (No. 12271511, No. 12201065, No. 12101105), the Austrian Science Fund (FWF) [Grant-DOI 10.55776/PAT1332123], and by the Natural Science Foundation of Fujian Province of China (No. 2024J01271). This work was also supported by the International Partnership Program of the Chinese Academy of Sciences (Grant No. 167GJHZ2023001FN).

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1. Introduction

Symbolic integration is a classical topic in computer algebra concerned with expressing indefinite integrals in closed form. Historical developments and fundamental results on this topic are surveyed in (Raab and Singer, 2022). From a computational perspective, the most systematic advances are due to Risch (1969, 1970) and numerous researchers who have refined, enhanced, extended, and implemented Risch's algorithm (Rothstein, 1976; Cherry, 1983; Trager, 1984; Davenport, 1986; Bronstein, 1990a,b, 2005; Raab, 2012).

For an elementary function, Risch's algorithm determines whether it has an elementary integral and computes such an integral if one exists.

Example 1.1. Let $f(x) = \frac{x \log(x)^3 + 1}{x \log(x)}$ and $g(x) = \frac{x \log(x)^3 + 1}{(x+3) \log(x)}$. By Risch's algorithm,

$$\int f dx = x \log(x)^2 - 2x \log(x) + 2x + \log(\log(x)) \quad \text{and} \quad \int g dx = \int \frac{x \log(x)^3 + 1}{(x+3) \log(x)} dx.$$

The algorithm yields an elementary integral of f and reveals that g has no elementary integral.

The most basic routine in Risch's algorithm is the Hermite-Ostrogradsky reduction for rational functions by Hermite (1872) and Ostrogradsky (1845). We refer to (Bronstein, 2005, Chapter 2) and (Geddes et al., 1992, Chapter 11) for modern presentations of this reduction.

In the rest of this section, let C be a field of characteristic zero.

Example 1.2. For every element $f \in C(x)$, the Hermite-Ostrogradsky reduction computes two elements $g, r \in C(x)$ such that $f = g' + r$, where $'$ stands for the usual derivation d/dx , and r is an x -proper fraction with a squarefree denominator. Moreover, r is unique, and $f \in C(x)'$ if and only if $r = 0$, where $C(x)'$ denotes the C -subspace consisting of derivatives in $C(x)$.

Set $S_x := \{s \in C(x) \mid s \text{ is } x\text{-proper and has a squarefree denominator}\}$. Then S_x is a C -subspace and $C(x) = C(x)' \oplus S_x$ by the Hermite-Ostrogradsky reduction. The projection from $C(x)$ to S_x with respect to the direct sum is a complete reduction for $C(x)'$ in the sense of Definition 5.67 in (Kauers, 2023), which is recalled below.

Definition 1.3. Let V be a C -linear space and U be a subspace. A linear operator $\Phi : V \rightarrow V$ is called a *complete reduction* for U if $v - \Phi(v) \in U$ for all $v \in V$ and $\ker(\Phi) = U$.

A straightforward linear algebra argument shows that a C -linear operator Φ on V is a complete reduction for U if and only if $V = U \oplus \text{im}(\Phi)$ and Φ is the projection from V onto $\text{im}(\Phi)$ with respect to this direct sum. In particular, all complete reductions are idempotent.

In the present paper, the subspace U in the above definition is typically the image of some linear operator on V . We therefore adopt the following definition for convenience.

Definition 1.4. Let V be a C -linear space and $\mathcal{L}$ be a linear operator on V . Another linear operator Φ on V is called a *complete reduction for the pair* $(V, \mathcal{L})$ if Φ is a complete reduction for $\text{im}(\mathcal{L})$. In other words, $V = \text{im}(\mathcal{L}) \oplus \text{im}(\Phi)$ and Φ is the projection from V onto $\text{im}(\Phi)$ with respect to this direct sum.

The next definition describes how a complete reduction decomposes an element additively.

Definition 1.5. Let Φ be a complete reduction for the pair $(V, \mathcal{L})$ and let $u, v \in V$ be such that

$$v = \mathcal{L}(u) + \Phi(v).$$

We call $\Phi(v)$ the *remainder of v with respect to Φ* , and $(u, \Phi(v))$ a *reduction pair of v with respect to Φ* (or an R -pair with respect to Φ for brevity).

Remark 1.6. Given a linear operator $\mathcal{L}$ on a linear space V , we say that *there is an algorithm for constructing a complete reduction for $(V, \mathcal{L})$* if we can fix a complement W of $\text{im}(\mathcal{L})$ in V and develop an algorithm that, given any $v \in V$, computes $u \in V$ and $w \in W$ such that $v = \mathcal{L}(u) + w$. Such an algorithm induces a complete reduction for $(V, \mathcal{L})$ by mapping v to w .

The Hermite-Ostrogradsky reduction in Example 1.2 not only induces a complete reduction for $(C(x), ')$, but also yields an algorithm for computing the corresponding R-pairs. Based on this reduction, an algorithm is developed to construct telescopers for bivariate rational functions in (Bostan et al., 2010). The algorithm separates the computation of telescopers from the computation of certificates. This is desirable in the typical situation where we are only interested in telescopers. This advantage has motivated the development of complete reductions in a variety of settings.

Example 1.7. Let h be a nonzero rational function in $C(x)$, and let V be the space spanned by the hyperexponential function $\exp(\int h dx)$ over $C(x)$. The Hermite reduction for hyperexponential functions developed in (Bostan et al., 2013) gives a complete reduction for $(V, d/dx)$.

Recall that a *derivation* $' : R \rightarrow R$ on a commutative ring R is an additive map satisfying Leibniz's rule: $(ab)' = a'b + ab'$ for all $a, b \in R$. The pair $(R, ')$ is called a *differential ring*; if R is a field, it is called a *differential field*. The set $\{c \in R \mid c' = 0\}$ forms a subring whose elements are called *constants*. This subring is a subfield if R is a field.

Let $(R, ')$ be a differential ring whose subring C of constants is a field. Then R is a C -linear space. Complete reductions have been established in various differential settings, including the field of differential fractions (Boulier et al., 2016), the field of algebraic functions (Chen et al., 2016), (sub)rings of D-finite functions (Chen et al., 2018b; van der Hoeven, 2021; Chen et al., 2023), exponential towers (Gao, 2024), and primitive towers (Du et al., 2025). A complete reduction for the pair $(C(x), \mathcal{L})$ is developed in (Bostan et al., 2018), where $\mathcal{L}$ denotes a linear differential operator. We refer to (Brochet and Salvy, 2024; Chen et al., 2025b,c) for complete reductions related to symbolic summation.

A complete reduction for $(R, ')$ decomposes $f \in R$ as $f = g' + r$ for some $g, r \in R$, where r is minimal with respect to a partial ordering defined in (Du et al., 2025, Section 1). In particular, f is a derivative in R if and only if $r = 0$. Thus, a complete reduction yields an additive decomposition described in (Chen et al., 2018a; Du et al., 2020). This offers an alternative and potentially more informative way to express integrals.

Example 1.8. By viewing the functions f and g in Example 1.1 as elements of the differential field $\mathbb{Q}(x, \log(x))$ with derivation $' = d/dx$, the complete reduction in (Du et al., 2025) yields

$$f(x) = h(x)' + \frac{1}{x \log(x)} \quad \text{and} \quad g(x) = h(x)' + \frac{1}{(x+3) \log(x)} - \frac{3 \log(x)^2}{x+3},$$

where $h(x) = x \log(x)^2 - 2x \log(x) + 2x$.

Note that $f(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder $1/(x \log(x))$, which has an elementary integral $\log(\log(x))$. By contrast, $g(x)$ is the sum of a derivative in $\mathbb{Q}(x, \log(x))$ and a remainder, which is not elementarily integrable. The elementary integrability of such remainders in logarithmic extensions can be determined by computing parametric logarithmic parts based on Theorem 5.3 in (Du et al., 2025). In contrast to Risch's algorithm as applied in Example 1.1, the complete reduction provides an in-field antiderivative $h(x)$ of the integrable part of $g(x)$, together with a minimal non-integrable part with respect to the partial ordering defined in (Du et al., 2025, Section 1). Both decompositions can also be obtained from the algorithms in (Chen et al., 2018a; Du et al., 2020).

We are interested in developing complete reductions for transcendental Liouvillian extensions. Such extensions are more general than transcendental elementary extensions (Bronstein, 2005, Definition 5.1.4) and do not contain any complicated reduced elements (Bronstein, 2005, Definition 3.5.2).

Definition 1.9. A differential field $F_n = C(t_1, \dots, t_n)$ with derivation $'$ is called a *transcendental Liouvillian extension* of C if

- (i) $t_1, \dots, t_n$ are algebraically independent over C ,
- (ii) for every $i \in \{1, \dots, n\}$, either $t'_i \in F_{i-1}$ or $t'_i/t_i \in F_{i-1}$, where $F_{i-1} = C(t_1, \dots, t_{i-1})$,
- (iii) C is the subfield of constants in F_n .

We call t_i a *primitive* (resp. *hyperexponential*) *generator* if $t'_i \in F_{i-1}$ (resp. $t'_i/t_i \in F_{i-1}$). The field F_n is called a *primitive* (resp. *hyperexponential*) *tower* if all the t_i are primitive (resp. hyperexponential).

Due to the presence of hyperexponential generators, a more general inductive framework is indispensable (see Example 3.1). A similar phenomenon also occurs in the inductive proof of the main theorem in (Risch, 1969). In fact, this theorem has two assertions. The first is of interest in elementary integration, while the second facilitates the proof. In the second assertion, the (parametric) Risch differential equation

$$y' + hy = \sum_{i=1}^m c_i g_i \quad (1)$$

first appeared in the literature. We write h here instead of the symbol f used in (Risch, 1969), because f often stands for an arbitrary element of a differential field in the sequel. In (1), h and the g_i belong to a differential field F , and the c_i are constants to be determined.

To develop complete reductions for symbolic integration, we define Risch operators using the homogeneous part of (1).

Definition 1.10. Let $(F, ')$ be a differential field and $h \in F$. The map

$$\begin{aligned} \mathcal{R}_h : F &\rightarrow F \\ y &\mapsto y' + hy \end{aligned}$$

is called the *Risch operator associated to h* .

Risch operators are linear over the subfield of constants in F . In particular, $\mathcal{R}_0$ is the derivation operator on F .

Assume that there is a complete reduction Φ_h for $(F, \mathcal{R}_h)$, where h is given on the left-hand side of (1). Then each of the g_i on the right-hand side of (1) has an R-pair $(p_i, \Phi_h(g_i))$ with respect to Φ_h . Therefore, (1) can be rewritten as $\mathcal{R}_h(y) = \sum_{i=1}^m c_i (\mathcal{R}_h(p_i) + \Phi_h(g_i))$, which is equivalent to

$$\mathcal{R}_h \left(y - \sum_{i=1}^m c_i p_i \right) = \sum_{i=1}^m c_i \Phi_h(g_i).$$

Since $\text{im}(\mathcal{R}_h) \cap \text{im}(\Phi_h) = \{0\}$, an element $y \in F$ and constants $c_1, \dots, c_m$ satisfy (1) if and only if $\sum_{i=1}^m c_i \Phi_h(g_i) = 0$ and $\mathcal{R}_h(y - \sum_{i=1}^m c_i p_i) = 0$. The first constraint leads to a linear system over the subfield C of constants in F , and the second amounts to determining the kernel of $\mathcal{R}_h$. In other words, the complete reduction finds all solutions of (1) in F by solving a linear system over C and the homogeneous Risch differential equation $y' + hy = 0$ in F .

Let F_n be a transcendental Liouvillian extension of C , and let h be any element of F_n . The main result of this paper is Theorem 7.1, which presents an algorithm for constructing a complete reduction for $(F_n, \mathcal{R}_h)$ for every $h \in F_n$. In doing so, we generalize the complete reduction for primitive towers presented in the Proceedings of ISSAC 2025 (Du et al., 2025) to general transcendental Liouvillian extensions. The generalization also includes the complete reduction in Example 1.7 and the one for exponential towers in (Gao, 2024) as special cases. In particular, setting $h := 0$ yields a complete reduction Φ_0 for derivatives in F_n . The new complete reduction enables us to determine the in-field integrability of functions in F_n directly.

Example 1.11. Let

$$f = \frac{x}{1 + \exp(x)} \cdot \exp\left(\frac{x}{1 + \exp(x)}\right).$$

It can be regarded as an element of the transcendental Liouvillian extension $\mathbb{C}(x, t, y)$ of $\mathbb{C}$, where $t = \exp(x)$ and $y = \exp\left(\frac{x}{1 + \exp(x)}\right)$. Indeed, $f = \frac{xy}{1+t} \in \mathbb{C}(x, t, y)$. We determine whether f has an integral in $\mathbb{C}(x, t, y)$.

The complete reduction given by the algorithm in Section 7 finds that $f = -\left(y + \frac{y}{t}\right)'$ and thus

$$\int f dx = -(1 + \exp(-x)) \cdot \exp\left(\frac{x}{1 + \exp(x)}\right).$$

The AXIOM-based computer algebra system FriCAS 1.3.13 returns the same integral (see FriCAS Team). However, the `int()` command (with the option `method=_RETURNVERBOSE`) in MAPLE 2026 leaves the integral unevaluated, as does the `Integrate[]` command in MATHEMATICA 14.3.

Computing this integral amounts to carefully handling a special case in Risch's algorithm, which corresponds to Proposition 6.13 (ii). Details are given in Example 7.4.

Raab (2012, Chapters 3 and 4) provides several building blocks that were either missing from or incomplete in the description of Risch's algorithm in (Bronstein, 2005) and presents a method for computing elementary integrals over admissible differential fields subject to some mild restrictions. Such fields are more general than transcendental Liouvillian extensions. Raab's method requires all the ingredients from Risch's algorithm for the transcendental case. The complete reduction presented in this paper leads to an alternative algorithm for elementary integration over transcendental Liouvillian extensions. It is based on the special structure of remainders described in Proposition 8.8 and the techniques in (Raab, 2012, Section 4.3.1) for dealing with the logarithmic derivative recognition problem (Bronstein, 2005, Section 5.12) and the parametric logarithmic derivative problem (Bronstein, 2005, Section 7.3).

Besides elementary integration, our complete reduction may also be used for reduction-based creative telescoping in transcendental Liouvillian extensions, as illustrated below.

Example 1.12. Let k be a positive integer,

$$R(k) = \int_0^{\frac{\pi}{2}} \cos(2kx) \log(\sin(x)) dx \quad \text{and} \quad I(k) = \int_0^{\frac{\pi}{2}} \sin(2kx) \log(\sin(x)) dx.$$

By the complete reduction, we find that $(k+1)S_k - k$ is a minimal telescoper for both $R(k)$ and $I(k)$, where S_k stands for the shift operator $k \mapsto k+1$. The telescoper yields the following two linear recurrences:

$$R(k+1) - \frac{k}{k+1}R(k) = 0 \quad \text{and} \quad I(k+1) - \frac{k}{k+1}I(k) = \frac{(-1)^k - 2k - 1}{4k(k+1)^2},$$

which allow us to express $R(k)$ and $I(k)$ in closed form. Details are given in Example 8.13.

Preliminary experiments in MAPLE 2026 indicate that our complete reduction outperforms the `int()` command in MAPLE 2026 for computing in-field integrals of derivatives of elements in certain transcendental elementary extensions. In particular, a significant speed-up is observed when integrands are derivatives of polynomials in the generators of such an extension (see Tables 5 and 7 in Section 9).

Let $F_n = C(t_1, \dots, t_{n-1}, t_n)$ be a transcendental Liouvillian extension of C . Our construction of a complete reduction proceeds by induction on n : the base case $n=0$ corresponds to the subfield of

constants, and for the inductive step we set $F := C(t_1, \dots, t_{n-1})$ and $t := t_n$. Then t is either primitive or hyperexponential over F .

Assume that there is an algorithm for constructing complete reductions for $(F, \mathcal{R}_\alpha)$ for all $\alpha \in F$. For an element $h \in F(t)$, the overall strategy for constructing a complete reduction for $(F(t), \mathcal{R}_h)$ is outlined in Fig. 1. Here, t -normalized elements, the companion operator $\mathcal{P}_h$, and echelon sequences are defined in Definitions 3.3, 3.5, and 2.10, respectively, while the differential subring $F\langle t \rangle$ of $F(t)$ is defined in (2).

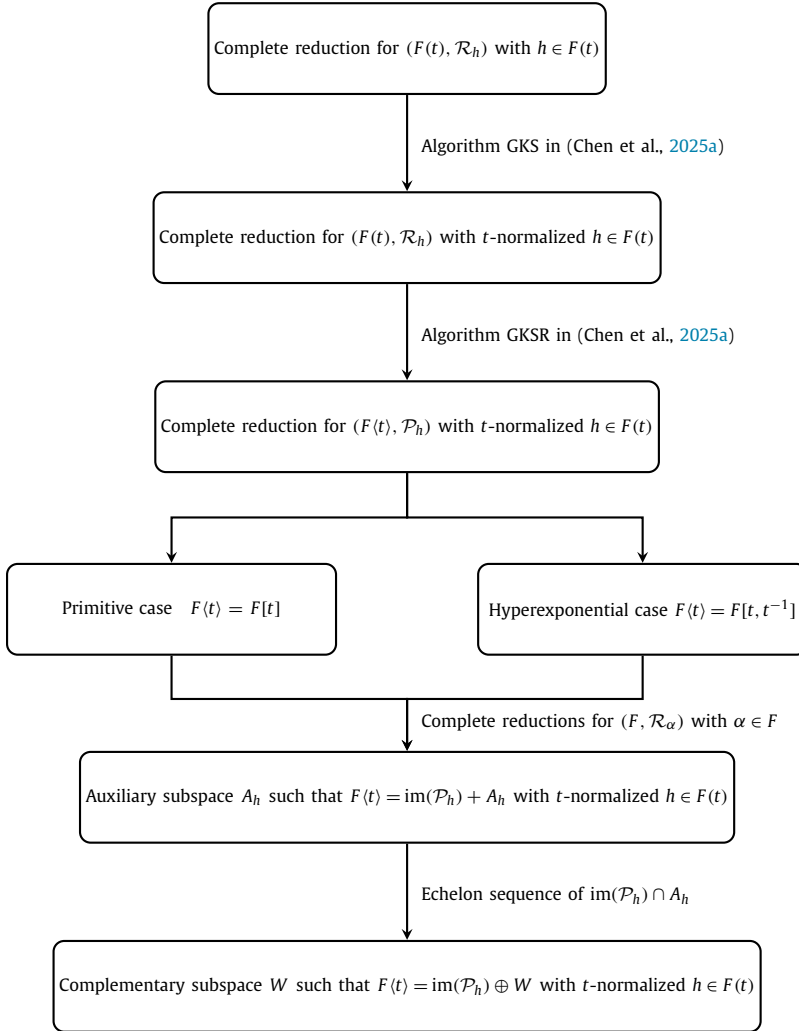


Fig. 1. General outline of the complete reduction construction.

We reduce the problem of constructing a complete reduction for $(F(t), \mathcal{R}_h)$ to the corresponding problem for $(F(t), \mathcal{P}_h)$ with t -normalized h . The idea behind such a transformation can be traced back to (Davenport, 1986, Condition b) on page 905) and (Bronstein, 2005, Section 6.1), in which the problem of solving Risch differential equations in $F(t)$ is reduced to the corresponding problem in the differential subring $F\langle t \rangle$. Similar transformations are given in (Geddes et al., 2004; Chen et al., 2025a), which can be viewed, respectively, as a differential analogue and a generalization of the minimal

decomposition for hypergeometric terms in (Abramov and Petkovšek, 2001, 2002). This step is based on Algorithms GKS and GKSR in (Chen et al., 2025a). Algorithm GKS decomposes an element of $F(t)$ as the sum of a t -normalized element and a logarithmic derivative, and Algorithm GKSR decomposes an element of $F(t)$ as the sum of an element of $\text{im}(\mathcal{R}_h)$, an element of $F(t)$, and a t -simple element (see Definition 2.1) under the assumption that $h \in F(t)$ is t -normalized.

Despite the differences between primitive and hyperexponential generators, we develop a unified approach to constructing a complete reduction for $(F(t), \mathcal{P}_h)$. Our approach constructs an auxiliary subspace A_h such that $F(t) = \text{im}(\mathcal{P}_h) + A_h$ and computes an echelon sequence of $\text{im}(\mathcal{P}_h) \cap A_h$. This sequence then enables us to find a complement of $\text{im}(\mathcal{P}_h)$ in a dual manner. Although related ideas have appeared in (Bostan et al., 2013, Section 4.1), (Gao, 2024, Chapter 4), and (Du et al., 2025, Section 3), a comprehensive generalization is needed in our setting because $F(t)$ is more involved than the differential fields considered in those works. This step makes essential use of two algorithms in (Raab, 2012, Section 4.3.1): one for recognizing logarithmic derivatives in F and the other for solving the parametric logarithmic derivative problem in F .

The remainder of this paper is organized as follows. Section 2 introduces basic terminology from symbolic integration and provides a dual description of complementary subspaces. In Section 3, we recall the notion of t -normalized elements, define companion operators, and reduce the problem of constructing complete reductions for a transcendental Liouvillian extension to the corresponding problem for a differential subring. Our inductive hypothesis is given in Section 4. The most technical inductive steps are treated for primitive generators in Section 5 and for hyperexponential generators in Section 6. The induction is completed in Section 7. Applications and computational benchmarks are presented in Sections 8 and 9, respectively. Section 10 contains concluding remarks. In the appendix, we describe algorithms derived from the constructive proofs given in Sections 5 and 6.

2. Preliminaries

This section consists of three parts. First, we introduce notation to be used in the sequel, along with fundamental terminology for symbolic integration. Next, we review several useful properties of residues relevant to elementary integration. Finally, we characterize certain complementary subspaces via kernels of linear functions. This characterization will enable us to construct complementary subspaces of infinite dimension in Sections 5 and 6.

2.1. Basic notation and terminology

In addition to the standard notation, we let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{N}^-$ denote the sets of positive, non-negative, and negative integers, respectively. Set $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$, and $[n]_0 := \{0, 1, \dots, n\}$ for $n \in \mathbb{N}_0$. In addition, we set $[0] := \emptyset$. For an abelian group $(G, +, 0)$, we write $G^\times$ for $G \setminus \{0\}$. Let R be a commutative ring, $S \subset R$, and $x \in R$. The set $\{sx \mid s \in S\}$ is denoted by either $S \cdot x$ or $x \cdot S$.

All fields appearing in the sequel are of characteristic zero. Let F be a field and $p \in F[t]^\times$. The degree and leading coefficient of p are denoted by $\deg_t(p)$ and $\text{lc}_t(p)$, respectively. In addition, $\deg_t(0) := -\infty$ and $\text{lc}_t(0) := 0$. We set $F[t]_{<d} := \{p \in F[t] \mid \deg_t(p) < d\}$ for all $d \in \mathbb{Z}$.

The ring of Laurent polynomials in t over F is denoted by $F[t, t^{-1}]$. Let $q \in F[t, t^{-1}]$ be of the form $q_k t^k + q_{k-1} t^{k-1} + \dots + q_{l+1} t^{l+1} + q_l t^l$, where $k, l \in \mathbb{Z}$, $k \geq l$, $q_k, q_{k-1}, \dots, q_{l+1}, q_l \in F$, and $q_k q_l \neq 0$. The *head degree*, *head coefficient*, *tail degree*, and *tail coefficient* of q are defined to be k , q_k , l , and q_l , and are denoted by $\text{hdeg}_t(q)$, $\text{hc}_t(q)$, $\text{tdeg}_t(q)$, and $\text{tc}_t(q)$, respectively. In addition, $\text{hdeg}_t(0) := -\infty$, $\text{hc}_t(0) := 0$, $\text{tdeg}_t(0) := \infty$, and $\text{tc}_t(0) := 0$. Since

$$F[t, t^{-1}] = F[t] \oplus (t^{-1} \cdot F[t^{-1}]),$$

we set q^+ and q^- to be the respective projections of q to $F[t]$ and $t^{-1} \cdot F[t^{-1}]$. For $Q \subset F[t, t^{-1}]$ and $l \in \mathbb{Z}$, we denote by $Q_{>l}$ the set of elements in Q whose tail degrees are greater than l .

For an element f of $F(t)$, its numerator and denominator are denoted by $\text{num}_t(f)$ and $\text{den}_t(f)$, respectively. Moreover, $\text{den}_t(f)$ is chosen to be monic. We say that f is *t-proper* if the degree of $\text{num}_t(f)$ is less than that of $\text{den}_t(f)$. For later convenience, the degree of f in t is defined to be the

maximum of the degrees of its numerator and denominator, and is denoted by $\text{Deg}_t(f)$. Note that $\text{deg}_t(0) = -\infty$ but $\text{Deg}_t(0) = 0$.

The subscript t will be omitted for brevity when the indeterminate t is clear from context.

Let $f \in F(t)^\times$, $a = \text{num}(f)$ and $b = \text{den}(f)$. For $p \in F[t]$ with $\text{deg}(p) > 0$, the *order of f at p* , denoted by $v_p(f)$, is defined as follows: if p occurs in b with multiplicity $m > 0$, then $v_p(f) = -m$; otherwise, $v_p(f)$ is equal to the multiplicity of p in a . The *order of f at infinity*, denoted by $v_\infty(f)$, is defined as $\text{deg}(b) - \text{deg}(a)$. By convention, we set $v_p(0) := \infty$ for all $p \in F[t]$ with $\text{deg}(p) > 0$, and $v_\infty(0) := \infty$.

Let $(F, ')$ be a differential field. The algebraic closure of F is denoted by $\bar{F}$. We set

$$F' := \{f' \mid f \in F\},$$

which is a linear space over the subfield of constants in F . An element of F is called a *logarithmic derivative* if it is equal to g'/g for some $g \in F^\times$. A differential field (E, δ) is called a *differential field extension* of $(F, ')$ if F is a subfield of E and $\delta|_F = '.$ The derivation δ will also be denoted by $'$ when no confusion can arise.

Let $(E, ')$ be a differential field extension of $(F, ')$ and $u \in E$. Then u is said to be *primitive* (resp. *hyperexponential*) over F if $u' \in F$ (resp. $u \neq 0$ and $u'/u \in F$). An element u of E is said to be *logarithmic* (resp. *exponential*) over F if u' is a logarithmic derivative in F (resp. $u \neq 0$ and $u'/u \in F'$). If u is logarithmic (resp. exponential) over F , then it is primitive (resp. hyperexponential) over F . The converses are false. For example, the logarithmic integral $\text{Li}(x)$, which equals $\int \frac{1}{\log x} dx$, is primitive but not logarithmic over $\mathbb{C}(x, \log(x))$, and $x \exp(x)$ is hyperexponential but not exponential over $\mathbb{C}(x)$.

An element t in a differential field extension of $(F, ')$ is called a *monomial over F* if t is transcendental over F and $t' \in F[t]$. A monomial t over F is said to be *regular* if $F(t)$ and F have the same subfield of constants. By Definition 1.9, a transcendental Liouvillian extension $C(t_1, \dots, t_n)$ is a differential field extension of C in which t_i is a regular monomial over $C(t_1, \dots, t_{i-1})$ and is either primitive or hyperexponential over the same field for all $i \in [n]$.

In the rest of this section, we let $(F, ')$ be a differential field, C be its subfield of constants, and t be a monomial over F . Then $F[t]$ is a differential subring of $F(t)$. A nonzero polynomial $p \in F[t]$ is said to be *normal* (resp. *special*) if $\gcd(p, p') = 1$ (resp. $\gcd(p, p') = p$) according to (Bronstein, 2005, Definition 3.4.2). The set of irreducible normal polynomials of positive degree is denoted by N_t .

Let us recall the notion of t -simple elements from (Chen et al., 2025a, Section 2.1).

Definition 2.1. An element of $F(t)$ is *t -simple* if it is t -proper and has a normal denominator.

All t -simple elements of $F(t)$ form an F -subspace, denoted by S_t . Let

$$F\langle t \rangle = \{f \in F(t) \mid \text{den}(f) \text{ is special}\}. \quad (2)$$

Then $F\langle t \rangle$ is a differential subring by Corollary 4.4.1 in (Bronstein, 2005). We remark that t -simple elements are not required to be t -proper by Definition 3.5.2 in (Bronstein, 2005). Our additional requirement of t -properness ensures that $S_t \cap F\langle t \rangle = \{0\}$, which is necessary for the direct sum (6) in Section 3.

In what follows, two distinct styles of lowercase Greek letters are used. Those set in the standard mathematical font, such as $\alpha, \beta, \gamma, \lambda, \mu, \sigma$, and τ , denote elements of $\bar{F}$. Those set in the Aegreya-LF style, such as ξ, η , and ω , denote elements of $F(t)$ that are not necessarily in $\bar{F}$.

Some basic facts about generators in a transcendental Liouvillian extension are collected in the following lemma for later reference.

Lemma 2.2. Let t be regular over F , $d \in \mathbb{Z}$, $\alpha \in F^\times$ and $p \in F[t]^\times$.

- (i) If $d > 0$ and t is primitive over F , then $(\alpha t^d)' = \alpha' t^d + (\alpha d t') t^{d-1}$, and its degree is d if α is not a constant and $d - 1$ otherwise.
- (ii) If $d \neq 0$ and t is hyperexponential over F , then $(\alpha t^d)' = (\alpha' + \alpha d(t'/t)) t^d$, and its head and tail degrees are both d .

- (iii) If t is primitive over F , then p is normal if and only if p is squarefree.
 (iv) If t is hyperexponential over F , then p is normal if and only if p is squarefree and $t \nmid p$.

Proof. (i) Let m be the degree of $(\alpha t^d)'$. By Leibniz's rule, $(\alpha t^d)' = \alpha' t^d + (\alpha d t') t^{d-1}$. Thus, $m = d$ if $\alpha' \neq 0$. Otherwise, $m = d - 1$ since t is both regular and primitive.

(ii) Similarly, Leibniz's rule implies that $(\alpha t^d)' = (\alpha' + \alpha d(t'/t)) t^d$, which is nonzero by Theorem 5.1.2 in (Bronstein, 2005). Since $t'/t \in F$, both head and tail degrees of $(\alpha t^d)'$ are equal to d .

(iii) and (iv) follow from Lemma 3.4.4, Theorems 5.1.1 and 5.1.2 in (Bronstein, 2005). $\square$

We use sequences in algorithmic descriptions to avoid cumbersome notation for list operations. Let $L : l_1, \dots, l_k$ be a sequence. Then l_i is denoted by $L[i]$ for all $i \in [k]$. The *length* of L is defined to be k and denoted by $\text{len}(L)$. In the description of an algorithm, comments are placed between $(*\dots*)$. When outlining algorithms, we abbreviate “with respect to” and “such that” as “w.r.t.” and “s.t.”, respectively.

2.2. Residues and elementary integration

We recall the definition of residues at elements of N_t from Definition 4.4.1 in (Bronstein, 2005).

Definition 2.3. Let $p \in N_t$ and $V_p = \{f \in F(t) \mid v_p(f) \geq -1\}$. For an element $f \in V_p$, the *residue of f at p* , denoted by $\text{residue}_p(f)$, is $\pi_p(fp/p')$, where π_p stands for the canonical homomorphism from the ring $\{q \in V_p \mid v_p(q) \geq 0\}$ to $F[t]/(p)$.

Note that $\text{residue}_p(f)$ is a congruence class. We identify $\text{residue}_p(f)$ with its unique representative of degree less than $\deg(p)$, because we are mainly concerned with constant residues.

Remark 2.4. The residue of $f \in S_t$ is well-defined at every $p \in N_t$, since $v_p(f) \geq -1$.

Two well-known facts about residues are:

- all residues of an element of $F[t]$ are zero;
- all residues of a logarithmic derivative in $F(t)$ are integers by Corollary 4.4.2 in (Bronstein, 2005).

The next two lemmas summarize some technical results scattered in (Du et al., 2023) that will be useful in proving Theorem 8.7.

Lemma 2.5. Let t be regular over F , and let $p \in F[t]^\times$ with $d = \deg(p)$ and $p_d = \text{lc}(p)$.

- (i) If t is primitive over F , then $p'/p = s + p'_d/p_d$ for some $s \in S_t$.
 (ii) If t is hyperexponential over F , then $p'/p = s + p'_d/p_d + dt'/t$ for some $s \in S_t$.
 (iii) All residues of the element s appearing in (i) or (ii) are integers.

Proof. (i) Let $q = p_d^{-1}p$. Then q is monic. By Exercise 3.1 in (Bronstein, 2005), $p'/p = p'_d/p_d + q'/q$. All irreducible factors of q are normal by Lemma 2.2 (iii). Applying Exercise 3.1 in (Bronstein, 2005) and Lemma 2.2 (i) again gives $q'/q \in S_t$. The conclusion follows by setting $s := q'/q$.

(ii) There exists an integer $m \in \mathbb{N}_0$ and a monic polynomial $r \in F[t]$ with $t \nmid r$ such that $p = p_d t^m r$. All irreducible factors of r are normal by Lemma 2.2 (iv). The logarithmic derivative identity implies

$$\frac{p'}{p} = \frac{p'_d}{p_d} + m \frac{t'}{t} + \frac{r'}{r}. \quad (3)$$

Set $s := r'/r - (d - m)t'/t$. Since $t'/t \in F$, it follows from Exercise 3.1 in (Bronstein, 2005) and Lemma 2.2 (ii) that $s \in S_t$. Substituting $dt'/t + s$ for $mt'/t + r'/r$ in (3) yields $p'/p = p'_d/p_d + dt'/t + s$.

(iii) The t -simple element s in (i) or (ii) is the difference of p'/p and an element of F . Since all residues of a logarithmic derivative are integers, so are the residues of s . $\square$

Recall that a differential field extension $F(u_1, \dots, u_m)$ is called an *elementary extension* of F if each u_j is either algebraic, logarithmic, or exponential over $F(u_1, \dots, u_{j-1})$ and the extension contains no new constants. We say that an element of F has an *elementary integral* over F if it is a derivative in some elementary extension of F by Definition 5.1.4 in (Bronstein, 2005).

Lemma 2.6. *Let t be regular over F and $f \in S_t^\times$. Assume that C is algebraically closed and that t is either primitive or hyperexponential over F . If all residues of f belong to C , then f has an elementary integral over $F(t)$.*

Proof. Let $\alpha_1, \dots, \alpha_k$ be the distinct roots of $\text{den}(f)$ in $\overline{F}$. Since all residues of f belong to C , it follows from Lemma 3.1 (i) in (Du et al., 2023) that

$$f = c_1 \frac{(t - \alpha_1)'}{t - \alpha_1} + \dots + c_k \frac{(t - \alpha_k)'}{t - \alpha_k} + p$$

for some $c_1, \dots, c_k \in C$ and $p \in F[t]$. If t is primitive over F , then $p = 0$ by $f \in S_t$ and Lemma 2.2 (i). If t is hyperexponential over F , then $p = ct'/t$ for some $c \in C$ by $f \in S_t$ and Lemma 2.2 (ii). Thus, f has an elementary integral over $F(t)$. $\square$

2.3. A dual description of complementary subspaces

Let V be a C -linear space. Every subspace of V can be written as the intersection of kernels of some linear functions on V .

For a nonempty subset S of V , $\text{span}_C(S)$ stands for the C -subspace spanned by S . The empty set spans the zero subspace $\{0\}$ by convention. Let Θ be a basis of V . For $\theta \in \Theta$, we denote by θ^* the linear function on V that maps θ to 1 and all other elements of Θ to 0.

Definition 2.7. Let V be a C -linear space, let Θ be a basis of V , let U be a subspace of V , and let $\mathbb{I} \subset \mathbb{N}_0$. A basis $\{u_i \mid i \in \mathbb{I}\}$ of U is called an *echelon basis of U with respect to Θ* if, for every $i \in \mathbb{I}$, there exists an element $\theta_i \in \Theta$ such that $u_i \notin \ker(\theta_i^*)$ and $u_j \in \ker(\theta_i^*)$ for all $j \in \mathbb{I}$ with $j < i$. In this case, we also call $\{u_i \mid i \in \mathbb{I}\}$ an *echelon basis of U with pivots $\{\theta_i \mid i \in \mathbb{I}\}$* .

An echelon basis $\{u_i \mid i \in \mathbb{N}_0\}$ of U with pivots $\{\theta_i \mid i \in \mathbb{N}_0\}$ is of the “upper triangular” form

$$\begin{aligned} & \vdots \\ u_i &= c_i \theta_i + \sum_{\theta \notin \{\theta_i, \theta_{i+1}, \dots\}} c_{i,\theta} \theta \\ & \vdots \\ u_1 &= c_1 \theta_1 + \sum_{\theta \notin \{\theta_1, \theta_2, \dots\}} c_{1,\theta} \theta \\ u_0 &= c_0 \theta_0 + \sum_{\theta \notin \{\theta_0, \theta_1, \dots\}} c_{0,\theta} \theta, \end{aligned}$$

where $i \in \mathbb{N}_0$, $c_i \in C^\times$, θ_i is the pivot of u_i , and $c_{i,\theta} \in C$, with only finitely many $c_{i,\theta}$ nonzero for each fixed i .

Lemma 2.8. *With the notation introduced in Definition 2.7, we further let $\{u_i \mid i \in \mathbb{I}\}$ be an echelon basis of U with pivots $\{\theta_i \mid i \in \mathbb{I}\}$. Then $V = U \oplus (\bigcap_{i \in \mathbb{I}} \ker(\theta_i^*))$.*

Proof. Let $\tilde{\Theta} = \Theta \setminus \{\theta_i \mid i \in \mathbb{I}\}$. Then $\cap_{i \in \mathbb{I}} \ker(\theta_i^*) = \text{span}_C \tilde{\Theta}$. The lemma follows from the observation that $\{u_i \mid i \in \mathbb{I}\} \cup \tilde{\Theta}$ is a basis of V . $\square$

When developing complete reductions, we face situations in which a subspace U of V is the image of some linear operator, and its echelon basis is difficult to construct directly. To construct a complement of U , we introduce an auxiliary subspace A such that $V = U + A$ and $U \cap A$ has an echelon basis. A complement of U can then be obtained from the following lemma.

Lemma 2.9. *With the notation introduced in Definition 2.7, we further let A be a subspace of V such that $V = U + A$. If $U \cap A$ has an echelon basis with pivots $\{\theta_i \mid i \in \mathbb{I}\}$, then the intersection*

$$A \cap \left(\bigcap_{i \in \mathbb{I}} \ker(\theta_i^*) \right)$$

is a complement of U in V .

Proof. Let $W = \cap_{i \in \mathbb{I}} \ker(\theta_i^*)$. Then $V = (U \cap A) \oplus W$ by Lemma 2.8. Since $U \cap A \subset A$, the modular law of subspaces gives $A = (U \cap A) + (W \cap A)$. Thus,

$$V = U + A = U + (U \cap A) + (W \cap A) = U + (W \cap A).$$

The conclusion follows immediately from $U \cap A \cap W = \{0\}$. $\square$

In this paper, reductions are typically carried out inside or modulo the image of a linear operator. This motivates the notion of echelon sequences, which allows us to find preimages and images simultaneously during reduction.

Definition 2.10. Let V be a C -linear space, let $\mathcal{L}$ be a linear operator on V , let Θ be a basis of V , and let U be a subspace contained in $\text{im}(\mathcal{L})$. Assume that $\{\mathcal{L}(v_i) \mid i \in \mathbb{I}\}$ is an echelon basis of U with pivots $\{\theta_i \mid i \in \mathbb{I}\}$. We call $\{(v_i, \mathcal{L}(v_i), \theta_i) \mid i \in \mathbb{I}\}$ an *echelon sequence of U with respect to $(V, \mathcal{L})$ and Θ* . Such a sequence is said to be *effective* if there is an algorithm that, given any $\theta \in \Theta$, decides whether $\theta = \theta_i$ for some $i \in \mathbb{I}$ and, if so, computes the index i and the corresponding triple $(v_i, \mathcal{L}(v_i), \theta_i)$.

Example 2.11. Let $V = C[x]$ with the C -basis $\Theta = \{x^i \mid i \in \mathbb{N}_0\}$, and let $\mathcal{L}$ be the linear operator that maps $p \in V$ to $(x^2 + 1)p$. Then $\text{im}(\mathcal{L})$ has an effective echelon sequence $\{(x^i, x^{i+2} + x^i, x^{i+2})\}_{i \in \mathbb{N}_0}$. By Lemma 2.8, a complement of $\text{im}(\mathcal{L})$ in V is $\cap_{i \in \mathbb{N}_0} \ker(\theta_i^*)$ with $\theta_i = x^{i+2}$. This complement is equal to $C[x]_{<2}$.

In the final part of this section, we prepare for the computation of R-pairs (see Definition 1.5). First, we recall the definition of effective bases from (Du et al., 2025, Section 2.2).

Definition 2.12. Let V be a C -linear space. A basis Θ of V is said to be *effective* if two algorithms are available: one takes as input any $v \in V^\times$ and computes an element $\theta \in \Theta$ such that $\theta^*(v) \neq 0$, and the other evaluates $\theta^*(v)$ for all $v \in V$ and $\theta \in \Theta$.

The transcendental Liouvillian extension $C(t_1, \dots, t_n)$ defined in Definition 1.9 admits an effective basis via successive irreducible partial fraction decompositions; details can be found in Algorithms 2.4, 2.6, and Remark 2.7 from (Du et al., 2025, Section 2.2). Moreover, echelon sequences constructed in Sections 5 and 6 are all effective.

Lemma 2.13. *With the notation introduced in Definition 2.10, we further assume that Θ is effective. Let*

$$E = \{(v_i, \mathcal{L}(v_i), \theta_i) \mid i \in \mathbb{I}\}$$

be an effective echelon sequence of U with respect to $(V, \mathcal{L})$ and Θ . Then for every $x \in V$, one can compute an element $v \in V$ and its image $\mathcal{L}(v) \in U$ (as a byproduct) such that $x - \mathcal{L}(v) \in \cap_{i \in \mathbb{I}} \ker(\theta_i^*)$.

Proof. Since Θ is effective, one can compute the finite expansion of x in the basis Θ . Since E is effective, one can construct the finite set Θ_x of all pivots occurring with nonzero coefficients in this expansion. If $\Theta_x = \emptyset$, set $v := 0$. Then $\mathcal{L}(v) = 0$, and the required property follows.

Assume that $\Theta_x \neq \emptyset$ and consider the maximal index j with $\theta_j \in \Theta_x$. Set $c_x := \theta_j^*(x)$ and $c_j := \theta_j^*(\mathcal{L}(v_j))$, which is nonzero. Define $y_j := c_x c_j^{-1} v_j$. Then $\mathcal{L}(y_j) = c_x c_j^{-1} \mathcal{L}(v_j) \in U$, and, by the maximality of j , we have $x - \mathcal{L}(y_j) \in \ker(\theta_i^*)$ for all $i \in \mathbb{I}$ with $i \geq j$.

Now replace x by $x - \mathcal{L}(y_j)$ and repeat the argument. Then the pivot set of the resulting x contains no θ_k with $k \geq j$. Thus, after finitely many iterations, we obtain a finite sum $v = \sum_l y_l$ and the corresponding $\mathcal{L}(v)$ with the required property. $\square$

3. Normalization and companion operators

Throughout this section, we let $(F, ')$ be a differential field, C be the subfield of constants in F , and t be a monomial over F .

We initially sought to construct a complete reduction for $(F(t), ')$ under the assumption that a complete reduction for $(F, ')$ was available. This assumption suffices when t is regular and primitive over F . However, it becomes insufficient when t is hyperexponential over F , as illustrated by the following example.

Example 3.1. Let t be both regular and hyperexponential. Set $\alpha = t'/t$, which belongs to F . For an element $\beta \in F$, we determine whether $\beta t \in F(t)'$. By an order argument, $\beta t \in F(t)'$ if and only if $\beta t \in F[t, t^{-1}]'$, which, by Lemma 2.2 (ii), is equivalent to $\beta t = (\gamma t)'$ for some $\gamma \in F$. In other words, γ is a solution of the Risch differential equation $y' + \alpha y = \beta$, or equivalently, $\beta \in \text{im}(\mathcal{R}_\alpha)$, where $\mathcal{R}_\alpha$ is given in Definition 1.10. Therefore, complete reductions for $(F(t), ')$ are inherently connected to Risch differential equations.

We will construct a complete reduction Φ_h for $(F(t), \mathcal{R}_h)$ for all $h \in F(t)$ under the assumption that a complete reduction for $(F, \mathcal{R}_\alpha)$ is available for every element $\alpha \in F$, where $\mathcal{R}_\alpha$ stands for the Risch operator on F associated to α .

Example 3.2. Let $F = C$ and $c \in C$. The Risch operator $\mathcal{R}_c : C \rightarrow C$ is given by $y \mapsto y' + cy$ for all $y \in C$. Then $\mathcal{R}_c(y) = cy$. It follows that $\text{im}(\mathcal{R}_c) = C$ if $c \neq 0$, and $\text{im}(\mathcal{R}_c) = \{0\}$ if $c = 0$. The complete reduction for $(C, \mathcal{R}_c)$ is the zero map if $c \neq 0$, and is the identity map if $c = 0$.

As in Theorem 6.1.1 from (Bronstein, 2005) for solving Risch equations, it suffices to focus on Risch operators associated to t -normalized elements defined below (Chen et al., 2025a, Section 3.2).

Definition 3.3. An element h of $F(t)$ is said to be t -normalized if, for all $i \in \mathbb{Z}$,

$$\gcd(\text{num}(h) - i \text{den}(h)', \text{den}(h)) = 1.$$

By Theorem 4.4.3 in (Bronstein, 2005), the gcd condition is equivalent to the requirement that no nonzero residue of h is an integer. All elements of F are t -normalized.

This normalization will simplify our construction in three ways:

- (i) It reduces the problem of constructing complete reductions for $(F(t), \mathcal{R}_h)$ with $h \in F(t)$ to that for $(F(t), \mathcal{R}_h)$ with t -normalized $h \in F(t)$ (see Proposition 3.4).
- (ii) It further transforms the problem into an equivalent one on $F\langle t \rangle$ (see Proposition 3.6).
- (iii) It ensures that $\mathcal{R}_h$ is injective if $h \in F(t) \setminus F$ and t is regular and either primitive or hyperexponential over F (see Lemma 4.3).

Following Davenport (1986) and Bronstein (2005, Definition 6.1.1), an element of $F(t)$ is said to be *weakly t -normalized* if it does not have any positive integer residue. Note that (i) and (ii) remain valid for weakly t -normalized elements, while (iii) relies on the assumption that h has no residue in $\mathbb{Z}^\times$.

For every element $h \in F(t)$, there exists a pair $(\xi, \eta) \in F(t) \times F(t)^\times$ such that ξ is t -normalized and $h = \xi + \eta'/\eta$. We call (ξ, η) a *normal form* of h . Normal forms are not unique. A canonical normal form can be computed by Algorithm GKS in (Chen et al., 2025a, Section 3.2).

Proposition 3.4. *Let $h \in F(t)$ with a normal form (ξ, η) . Then the following assertions hold.*

- (i) $\mathcal{R}_h = \eta^{-1} \circ \mathcal{R}_\xi \circ \eta$, where η is understood as the multiplication map $f \mapsto \eta f$ for all $f \in F(t)$.
- (ii) Assume further that Φ_ξ is a complete reduction for $(F(t), \mathcal{R}_\xi)$. Then $\Phi_h := \eta^{-1} \circ \Phi_\xi \circ \eta$ is a complete reduction for $(F(t), \mathcal{R}_h)$.
- (iii) For all $f \in F(t)$, $(\eta^{-1}g, \eta^{-1}r)$ is an R -pair of f with respect to Φ_h if (g, r) is an R -pair of ηf with respect to Φ_ξ .

Proof. (i) For all $f \in F(t)$, $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \eta^{-1}((\eta f)' + \xi(\eta f))$. A straightforward calculation using $h = \xi + \eta'/\eta$ then gives $\eta^{-1} \circ \mathcal{R}_\xi \circ \eta(f) = \mathcal{R}_h(f)$.

(ii) Since Φ_ξ is idempotent, so is Φ_h . Hence, $\text{im}(\Phi_h)$ is a complement of its kernel in $F(t)$. It suffices to verify that $\ker(\Phi_h) = \text{im}(\mathcal{R}_h)$. By (i), $\text{im}(\mathcal{R}_h) = \eta^{-1} \cdot \text{im}(\mathcal{R}_\xi)$. From $\Phi_h = \eta^{-1} \circ \Phi_\xi \circ \eta$, we have $\ker(\Phi_h) = \eta^{-1} \cdot \ker(\Phi_\xi)$. Thus, $\ker(\Phi_h) = \text{im}(\mathcal{R}_h)$ by $\ker(\Phi_\xi) = \text{im}(\mathcal{R}_\xi)$.

(iii) Let (g, r) be an R -pair of ηf with respect to Φ_ξ . Then $\eta f = \mathcal{R}_\xi(g) + r$ and $r \in \text{im}(\Phi_\xi)$. It follows that $f = \mathcal{R}_h(\eta^{-1}g) + \eta^{-1}r$, which, together with $\eta^{-1}r \in \text{im}(\Phi_h)$, implies that $(\eta^{-1}g, \eta^{-1}r)$ is an R -pair of f with respect to Φ_h . $\square$

In the rest of this section, we let $h \in F(t)$ be t -normalized and set $a := \text{num}(h)$, $b := \text{den}(h)$, and

$$S_{t,h} := \{s \in S_t \mid \gcd(\text{den}(s), b) = 1\}, \quad (4)$$

which is also a C -subspace of $F(t)$. For all $f \in F(t)$, Algorithm GKSR in (Chen et al., 2025a, Section 3.4) computes a triple $(g, r, s) \in F(t) \times F(t) \times S_{t,h}$ such that

$$f = \mathcal{R}_h(g) + \frac{r}{b} + s. \quad (5)$$

Moreover, s in (5) is unique. Therefore,

$$F(t) = (\text{im}(\mathcal{R}_h) + b^{-1} \cdot F(t)) \oplus S_{t,h}. \quad (6)$$

This direct sum generalizes the splitting $F(t) = (F(t)' + F(t)) \oplus S_t$ induced by Algorithm HERMITEREDUCE in (Bronstein, 2005, Section 5.6). It remains to find a C -subspace $W \subset F(t)$ with

$$\text{im}(\mathcal{R}_h) + (b^{-1} \cdot F(t)) = \text{im}(\mathcal{R}_h) \oplus (b^{-1} \cdot W).$$

In the special case $F(t) = C(x)$, such a subspace can be constructed via polynomial reduction in (Bostan et al., 2013, Section 4.1). This motivates us to introduce the notion of companion operators.

Definition 3.5. The map

$$\begin{array}{ccc} \mathcal{P}_h : F(t) & \rightarrow & F(t) \\ r & \mapsto & br' + ar \end{array}$$

is called the *companion operator* of $\mathcal{R}_h$.

The following proposition allows us to reduce the problem of constructing a complete reduction for $(F(t), \mathcal{R}_h)$ to the corresponding problem for $(F(t), \mathcal{P}_h)$.

Proposition 3.6. *If W is a complement of $\text{im}(\mathcal{P}_h)$ in $F\langle t \rangle$, then*

$$F(t) = \text{im}(\mathcal{R}_h) \oplus (b^{-1} \cdot W \oplus S_{t,h}). \quad (7)$$

Proof. By (6), $F(t) = (\text{im}(\mathcal{R}_h) + b^{-1} \cdot \text{im}(\mathcal{P}_h) + b^{-1} \cdot W) \oplus S_{t,h}$. Then

$$F(t) = (\text{im}(\mathcal{R}_h) + b^{-1} \cdot W) \oplus S_{t,h}$$

by $b^{-1} \cdot \text{im}(\mathcal{P}_h) \subset \text{im}(\mathcal{R}_h)$.

It remains to show that $\text{im}(\mathcal{R}_h) \cap (b^{-1} \cdot W) = \{0\}$. Assume that $v \in \text{im}(\mathcal{R}_h) \cap (b^{-1} \cdot W)$. Then $v = w/b \in \text{im}(\mathcal{R}_h)$ for some $w \in W$, and thus there exists $g \in F(t)$ such that $bg' + ag = w$. Since h is t -normalized, we have $g \in F\langle t \rangle$ by Theorem 6.1.2 in (Bronstein, 2005) or Theorem 3.12 (iii) in (Chen et al., 2025a). Hence, $w \in \text{im}(\mathcal{P}_h)$. Since $\text{im}(\mathcal{P}_h) \cap W = \{0\}$, we have $w = 0$, and hence $v = 0$. $\square$

Remark 3.7. With the notation of Proposition 3.6, we further let $\Phi_{h,W}$ be the projection from $F(t)$ to $(b^{-1} \cdot W \oplus S_{t,h})$ with respect to (7), and $\Psi_{h,W}$ be the projection from $F\langle t \rangle$ onto W with respect to $F(t) = \text{im}(\mathcal{P}_h) \oplus W$. Then $\Phi_{h,W}$ and $\Psi_{h,W}$ are respective complete reductions for $(F(t), \mathcal{R}_h)$ and $(F\langle t \rangle, \mathcal{P}_h)$.

Assume that one can compute R-pairs with respect to $\Psi_{h,W}$. Then R-pairs with respect to $\Phi_{h,W}$ can be computed as follows. Let $f \in F(t)$.

1. Find $(g, r, s) \in F(t) \times F\langle t \rangle \times S_{t,h}$ satisfying (5) by Algorithm GKSR in (Chen et al., 2025a, Section 3.4).
2. Compute an R-pair $(u, v) \in F\langle t \rangle \times W$ of r with respect to $\Psi_{h,W}$.

Then the pair $(g + u, b^{-1}v + s)$ is an R-pair of f with respect to $\Phi_{h,W}$. This follows from the fact that $b^{-1}v + s \in (b^{-1} \cdot W) \oplus S_{t,h}$ and from the calculation

$$\begin{aligned} f &= \mathcal{R}_h(g) + b^{-1}r + s && \text{(by (5))} \\ &= \mathcal{R}_h(g) + b^{-1}(\mathcal{P}_h(u) + v) + s && \text{(by } r = \mathcal{P}_h(u) + v) \\ &= \mathcal{R}_h(g) + \mathcal{R}_h(u) + b^{-1}v + s && \text{(by } \mathcal{R}_h(u) = b^{-1}\mathcal{P}_h(u)) \\ &= \mathcal{R}_h(g + u) + b^{-1}v + s. \end{aligned}$$

Remark 3.8. If $h = 0$, then $b = 1$, and (7) in Proposition 3.6 simplifies to

$$F(t) = F(t)' \oplus (W \oplus S_t),$$

which leads to a complete reduction for $(F(t), ')$.

4. Inductive hypothesis and notational convention

Having reduced the problem of constructing complete reductions to the setting of differential subrings, we now formulate the inductive framework for our construction.

In this section, let $n \in \mathbb{N}$ and let $C(t_1, \dots, t_n)$ be a transcendental Liouvillian extension of C . For inductive purposes, we set

$$F := C(t_1, \dots, t_{n-1}) \quad \text{and} \quad t := t_n.$$

By equation (5.1) in (Bronstein, 2005), the differential ring $F\langle t \rangle$ takes the form $F[t]$ when t is primitive, and $F[t, t^{-1}]$ when t is hyperexponential. This dichotomy structures our inductive arguments in Sections 5 and 6.

We now state the inductive hypothesis used throughout the remainder of the construction.

Hypothesis 4.1. Assume that there is a map assigning to each $\alpha \in F$ a complete reduction Φ_α for $(F, \mathcal{R}_\alpha)$, and that, for every $\alpha, \beta \in F$, an R-pair of β with respect to Φ_α can be computed algorithmically.

By the hypothesis, for every $\alpha \in F$, $\text{im}(\mathcal{R}_\alpha) \cap \text{im}(\Phi_\alpha) = \{0\}$, and every element $\beta \in F$ has an R-pair $(\gamma, \Phi_\alpha(\beta)) \in F^2$ with respect to Φ_α ; that is,

$$\beta = \mathcal{R}_\alpha(\gamma) + \Phi_\alpha(\beta)$$

for some $\gamma \in F$.

The base case of our induction is given by the following example.

Example 4.2. Let $F = \mathbb{C}$. By Example 3.2, the map in Hypothesis 4.1 assigns the identity operator to 0 and the zero operator to every element of $\mathbb{C}^\times$.

By Propositions 3.4 and 3.6, it suffices to construct complete reductions for $(F\langle t \rangle, \mathcal{P}_h)$ with t -normalized h . Let $h \in F\langle t \rangle$ be t -normalized. In the next two sections, we will construct a complete reduction for $(F\langle t \rangle, \mathcal{P}_h)$ in three steps:

1. Construct a subspace $A_h \subset F\langle t \rangle$ such that $F\langle t \rangle = \text{im}(\mathcal{P}_h) + A_h$ using (10), (26), and (27).
2. Construct an echelon sequence E of $\text{im}(\mathcal{P}_h) \cap A_h$ using Corollaries 5.17 and 5.18 in the primitive case and Proposition 6.14 in the hyperexponential case.
3. Determine a complement W of $\text{im}(\mathcal{P}_h)$ in $F\langle t \rangle$ using E and Lemma 2.9.

Once these steps are completed, we obtain the direct sum decomposition $F\langle t \rangle = \text{im}(\mathcal{P}_h) \oplus W$, and the projection from $F\langle t \rangle$ onto W gives the desired complete reduction. These three steps were carried out in (Du et al., 2025, Section 3) for the case where t is primitive and $h = 0$. However, additional technical obstacles emerge when $h \neq 0$ or t is hyperexponential.

The notion of auxiliary subspaces arises from our attempt to simplify the coefficients of elements in $F\langle t \rangle$ modulo $\text{im}(\mathcal{P}_h)$. Such a simplification relies on an elaborate application of integration by parts in the case $h = 0$ (Chen et al., 2018a; Du et al., 2020).

A basis of $\text{im}(\mathcal{P}_h) \cap A_h$ is constructed using two algorithms presented in (Raab, 2012, Section 4.3.1): one for recognizing logarithmic derivatives in F and the other for solving the parametric logarithmic derivative problem. Note that these two problems can be solved algorithmically for transcendental Liouvillian extensions in (Raab, 2012, Section 4.3.1), whereas their algorithmic solvability is not guaranteed for general differential fields. Although such a basis may be infinite when t is primitive, its regular form described in Proposition 5.15 allows us to construct an echelon sequence of the intersection. By Lemma 2.9, this sequence induces a complement of $\text{im}(\mathcal{P}_h)$ in $F\langle t \rangle$, and, by Lemma 2.13, it leads to an algorithm for computing R-pairs.

The following injectivity result allows us to transform a $\mathbb{C}$ -basis of a subspace of $F\langle t \rangle$ into a $\mathbb{C}$ -basis of its image under $\mathcal{P}_h$.

Lemma 4.3. If $h \in F\langle t \rangle \setminus F$ is t -normalized, then both $\mathcal{R}_h$ and $\mathcal{P}_h$ are injective.

Proof. Suppose that $u \in \ker(\mathcal{R}_h)^\times$. Then $h = -u'/u$ by Definition 1.10. The logarithmic derivative identity implies that h is a $\mathbb{Z}$ -linear combination of logarithmic derivatives of polynomials in $F[t]$. Since h is t -normalized, no logarithmic derivative of an element of N_t can occur with a nonzero coefficient in this $\mathbb{Z}$ -linear combination. Since t is either primitive or hyperexponential over F , it follows that $h \in F$, a contradiction. Thus, $\mathcal{R}_h$ is injective. Consequently, $\mathcal{P}_h$ is injective, since it is the restriction of $\text{den}(h)\mathcal{R}_h$ to $F\langle t \rangle$. $\square$

As the final preparation, we adopt a notational convention for the next two sections.

Convention 4.4.

(i) Let $C(t_1, \dots, t_n)$ be a transcendental Liouvillian extension of C . Set

$$F := C(t_1, \dots, t_{n-1}) \quad \text{and} \quad t := t_n.$$

(ii) Assume that t is primitive over F in Section 5 and hyperexponential over F in Section 6.

(iii) Let h be a t -normalized element of $F(t)$.

(iv) Set $a := \text{num}(h)$, $b := \text{den}(h)$, and $m := \text{Deg}(h)$. Write

$$a = a_m t^m + a_{m-1} t^{m-1} + \dots + a_0 \quad \text{and} \quad b = b_m t^m + b_{m-1} t^{m-1} + \dots + b_0,$$

where $a_i, b_i \in F$ and $b_m = 1$ whenever $b_m \neq 0$.

5. The primitive case

By Convention 4.4 (ii), t is primitive throughout this section. Then $F(t) = F[t]$. The coefficients a_m, b_m, a_{m-1} and b_{m-1} in Convention 4.4 (iv) will play a significant role in reducing polynomials. Note that both a_{m-1} and b_{m-1} are zero if $m = 0$.

To describe the leading terms of elements in $\text{im}(\mathcal{P}_h)$, we define a C -linear operator

$$\begin{aligned} \mathcal{L}: F &\rightarrow F \\ \alpha &\mapsto b_m \alpha' + a_m \alpha. \end{aligned} \tag{8}$$

Then, for an element $f \in F[t]^\times$ with $d = \deg(f)$ and $f_d = \text{lc}(f)$, we have

$$\mathcal{P}_h(f) \equiv \mathcal{L}(f_d) t^{m+d} \pmod{F[t]_{<m+d}} \tag{9}$$

since $\mathcal{P}_h(f) \equiv (b_m f'_d) t^{m+d} + (a_m f_d) t^{m+d} \pmod{F[t]_{<m+d}}$.

Remark 5.1. By Convention 4.4 and (8), the following two assertions hold.

- (i) If $v_\infty(h) < 0$, then $a_m \neq 0$ and $b_m = 0$. Consequently, $\mathcal{L}$ is injective.
- (ii) If $v_\infty(h) \geq 0$, then $b_m = 1$ and $\mathcal{L} = \mathcal{R}_{a_m}$, which is the Risch operator on F associated to a_m .

The next lemma will be used frequently for constructing echelon sequences in this section.

Lemma 5.2. If $\lambda \in \ker(\mathcal{L})^\times$, then $\Phi_{a_m}(\lambda t') \neq 0$, where Φ_{a_m} is the complete reduction for $(F, \mathcal{R}_{a_m})$ given in Hypothesis 4.1.

Proof. Since $\ker(\mathcal{L}) \neq \{0\}$, we have $v_\infty(h) \geq 0$ and hence $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1. It follows that $\lambda \in \ker(\mathcal{R}_{a_m})^\times$. Thus, $\lambda' + a_m \lambda = 0$. Suppose that $\Phi_{a_m}(\lambda t') = 0$. Then $\lambda t' \in \text{im}(\mathcal{R}_{a_m})$. Hence, there exists $\mu \in F$ such that $\mu' + a_m \mu = \lambda t'$. The two equalities involving a_m imply that $t' = (\mu/\lambda)'$, which contradicts the regularity of t . $\square$

The rest of this section has three parts. In Section 5.1, we first define the auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ and develop a corresponding reduction from $F[t]$ to U_h modulo $\text{im}(\mathcal{P}_h)$. Section 5.2 presents a criterion to determine whether $\text{im}(\mathcal{P}_h) \cap U_h = \{0\}$. An echelon sequence of this intersection, together with a complete reduction for $(F(t), \mathcal{R}_h)$, is constructed in Section 5.3.

5.1. Auxiliary subspaces

The idea of auxiliary subspaces arises from the attempt to decompose an element $f \in F[t]$ as $g' + r$, where $' = \mathcal{R}_0$ is the derivation on $F[t]$, $g, r \in F[t]$, $\deg(r) \leq \deg(f)$, and all coefficients of r lie

in $\text{im}(\Phi_0)$. Note that Φ_0 is the complete reduction for $(F, ')$ given in Hypothesis 4.1. Such a decomposition is always possible by Lemma 3.1 in (Du et al., 2025).

We generalize this idea from $'$ to $\mathcal{R}_h$, where h is given in Convention 4.4 (iii) and (iv).

To this end, we let $f \in F[t]$, $d = \deg(f)$ and $f_d = \text{lc}(f)$, and assume that $d \geq m$. Then

$$f \equiv \begin{cases} \mathcal{P}_h(a_m^{-1} f_d t^{d-m}) \mod F[t]_{<d} & \text{if } v_\infty(h) < 0, \\ \mathcal{P}_h(\alpha t^{d-m}) + \beta t^d \mod F[t]_{<d} & \text{if } v_\infty(h) \geq 0, \end{cases} \quad (10)$$

where $(\alpha, \beta) \in F^2$ is an R-pair of f_d with respect to Φ_{a_m} given in Hypothesis 4.1.

The first congruence holds by expanding $\mathcal{P}_h(a_m^{-1} f_d t^{d-m})$ directly modulo $F[t]_{<d}$ and noting that $b_m = 0$ since $v_\infty(h) < 0$. To show the second congruence, we note that $f_d = \mathcal{R}_{a_m}(\alpha) + \beta$ by Definition 1.5. Then $f_d = \mathcal{L}(\alpha) + \beta$ by Remark 5.1 (ii). So $f \equiv \mathcal{L}(\alpha) t^d + \beta t^d \mod F[t]_{<d}$. On the other hand, $\mathcal{P}_h(\alpha t^{d-m}) \equiv \mathcal{L}(\alpha) t^d \mod F[t]_{<d}$ by (9). Combining the last two congruences proves the second congruence in (10).

Iteratively applying the two congruences in (10) to elements of $F[t]_{<d}$, we arrive at the following definition.

Definition 5.3. The auxiliary subspace associated to $(F[t], \mathcal{P}_h)$ is defined as

$$U_h := \begin{cases} F[t]_{<m} & \text{if } v_\infty(h) < 0, \\ F[t]_{<m} + \sum_{i \in \mathbb{N}_0} \text{im}(\Phi_{a_m}) \cdot t^{m+i} & \text{if } v_\infty(h) \geq 0. \end{cases}$$

The set U_h is a C -subspace of $F[t]$ because both $F[t]_{<m}$ and $\text{im}(\Phi_{a_m})$ are C -subspaces of $F[t]$. The above definition generalizes the notion of auxiliary subspaces in (Du et al., 2025), which corresponds to the case $m = 0$ and $a_m = 0$.

Example 5.4. Let $F = C$, $t = x$ and $' = d/dx$. By Convention 4.4,

	a	b	m	a_m	b_m	$v_\infty(h)$
$h = 0$	0	1	0	0	1	∞
$h = x$	x	1	1	1	0	-1

It follows from Example 4.2 that the auxiliary subspaces associated to $(F[t], \mathcal{P}_0)$ and $(F[t], \mathcal{P}_x)$ are $C[x]$ and C , respectively.

Proposition 5.5. Let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and $f \in F[t]^\times$ with $d = \deg(f)$. Then there exists a pair $(g, r) \in F[t] \times U_h$ with $\deg(g) \leq d - m$ and $\deg(r) \leq d$ such that $f = \mathcal{P}_h(g) + r$. Consequently, $F[t] = \text{im}(\mathcal{P}_h) + U_h$.

Proof. If $d < m$, then $f \in U_h$. So it suffices to set $g := 0$ and $r := f$. Assume that $d \geq m$, and that the conclusion holds for polynomials of degrees lower than d . Let $f_d = \text{lc}(f)$.

If $v_\infty(h) < 0$, then $f \equiv \mathcal{P}_h(a_m^{-1} f_d t^{d-m}) \mod F[t]_{<d}$ by (10). Applying the inductive hypothesis to the resulting polynomial of degree less than d , we obtain the desired conclusion.

Otherwise, $v_\infty(h) \geq 0$. Let (α, β) be an R-pair of f_d with respect to Φ_{a_m} . Then

$$f \equiv \mathcal{P}_h(\alpha t^{d-m}) \mod U_h + F[t]_{<d}$$

by the second congruence in (10) and $\beta \in \text{im}(\Phi_{a_m})$. It follows from the inductive hypothesis and $F[t]_{<m} \subset U_h$ that $f \equiv \mathcal{P}_h(g) \mod U_h$ for some $g \in F[t]$ with $\deg(g) \leq d - m$. Furthermore, $\deg(\mathcal{P}_h(g)) \leq d$ by (9). The conclusion holds by setting $r := f - \mathcal{P}_h(g)$. $\square$

Definition 5.6. Let f and (g, r) be given in Proposition 5.5. We call (g, r) an *auxiliary pair* of f with respect to $(F[t], \mathcal{P}_h)$, or an *auxiliary pair* of f if $(F[t], \mathcal{P}_h)$ is clear from context. In addition, we define $(0, 0)$ to be the auxiliary pair of 0.

The proof of Proposition 5.5 leads to Algorithm A.1 in Section A.1. The next corollary reveals a relation between auxiliary pairs and R-pairs in the special case $h \in F$.

Corollary 5.7. Let $h \in F$ and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then, for every element $f \in F^\times$, an auxiliary pair of f with respect to $(F[t], \mathcal{P}_h)$ is an R-pair of f with respect to Φ_h .

Proof. Since $f \in F$, its degree is zero. So every auxiliary pair (α, β) of f with respect to $(F[t], \mathcal{P}_h)$ belongs to F^2 by Proposition 5.5. It remains to show that (α, β) is an R-pair of f with respect to Φ_h . By Convention 4.4, we have

$$\begin{array}{ccccccc} & a & b & m & a_m & b_m & v_\infty(h) \\ \hline h \in F & h & 1 & 0 & h & 1 & \geq 0 \end{array}$$

Then the auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ is $\sum_{i \in \mathbb{N}_0} \text{im}(\Phi_h) \cdot t^i$ by Definition 5.3. Thus, $\beta \in \text{im}(\Phi_h)$ since $\beta \in U_h \cap F$. On the other hand, the restrictions of $\mathcal{R}_h$ and $\mathcal{P}_h$ to F are equal by Definitions 1.10 and 3.5. Therefore, $f = \mathcal{P}_h(\alpha) + \beta = \mathcal{R}_h(\alpha) + \beta$. It follows from $F = \text{im}(\mathcal{R}_h) \oplus \text{im}(\Phi_h)$ that $\beta = \Phi_h(f)$, that is, (α, β) is an R-pair with respect to Φ_h . $\square$

5.2. Intersecting $\text{im}(\mathcal{P}_h)$ with the associated auxiliary subspace

In this subsection, we let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$. By Proposition 5.5, $F[t] = \text{im}(\mathcal{P}_h) + U_h$. Then $F[t] = \text{im}(\mathcal{P}_h) \oplus U_h$ if and only if $\text{im}(\mathcal{P}_h) \cap U_h = \{0\}$. Setting $I_h := \text{im}(\mathcal{P}_h) \cap U_h$, we aim to derive a necessary and sufficient condition for $I_h = \{0\}$.

The following lemma describes the leading coefficients of nonzero elements of $\mathcal{P}_h^{-1}(U_h)$.

Lemma 5.8. If $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]^\times$, then $\text{lc}(f) \in \ker(\mathcal{L})$.

Proof. Let $d = \deg(f)$ and $f_d = \text{lc}(f)$.

If $v_\infty(h) < 0$, then $U_h = F[t]_{<m}$ by Definition 5.3. So the degree of $\mathcal{P}_h(f)$ is less than m . Consequently, $\mathcal{L}(f_d) = 0$ because of (9).

Assume that $v_\infty(h) \geq 0$. Since $\mathcal{P}_h(f) \in U_h$, it follows from (9) and Definition 5.3 that

$$\mathcal{L}(f_d) \in \text{im}(\Phi_{a_m}).$$

Since $\mathcal{L} = \mathcal{R}_{a_m}$ by Remark 5.1 (ii) and $\text{im}(\mathcal{R}_{a_m}) \cap \text{im}(\Phi_{a_m}) = \{0\}$, we have $\mathcal{L}(f_d) = 0$. $\square$

The next proposition is a criterion for $I_h = \{0\}$.

Proposition 5.9. $I_h = \{0\}$ if and only if $\ker(\mathcal{L}) = \{0\}$. In particular, $I_h = \{0\}$ if $v_\infty(h) < 0$.

Proof. Assume that $\ker(\mathcal{L}) = \{0\}$. Let $\mathcal{P}_h(f) \in U_h$ for some $f \in F[t]$. Suppose $f \neq 0$. Then $\text{lc}(f)$ belongs to $\ker(\mathcal{L})$ by Lemma 5.8, which contradicts $\ker(\mathcal{L}) = \{0\}$. Hence, $I_h = \{0\}$.

Conversely, assume that $I_h = \{0\}$, and let $\lambda \in \ker(\mathcal{L})$. If $\lambda = 0$, there is nothing to prove. Assume that $\lambda \neq 0$. By (9), $\mathcal{P}_h(\lambda) \in F[t]_{<m}$. Then $\mathcal{P}_h(\lambda) \in U_h$ by Definition 5.3. Consequently, $\mathcal{P}_h(\lambda) = 0$ by $I_h = \{0\}$. If $h \in F(t) \setminus F$, then $\lambda = 0$ by Lemma 4.3, a contradiction.

Assume that $h \in F$. Then $m = 0$, $a = a_m = h$ and $b = b_m = 1$ by Convention 4.4. So $\mathcal{P}_h(\lambda t) = \lambda t'$ by $\lambda \in \ker(\mathcal{L})$. Since $\lambda t' \in F$, it has an auxiliary pair of the form $(\mu, \Phi_h(\lambda t'))$ for some $\mu \in F$ by Corollary 5.7, where Φ_h is the complete reduction for $(F, \mathcal{R}_h)$ given in Hypothesis 4.1. Thus,

$$\mathcal{P}_h(\lambda t) = \lambda t' = \mathcal{P}_h(\mu) + \Phi_h(\lambda t').$$

Consequently, $\mathcal{P}_h(\lambda t - \mu) = \Phi_h(\lambda t')$, which, together with $\Phi_h(\lambda t') \in U_h$, implies that $\Phi_h(\lambda t') \in I_h$. We conclude from $I_h = \{0\}$ that $\Phi_h(\lambda t') = 0$. Then $\lambda = 0$ by Lemma 5.2, a contradiction. Therefore, no nonzero $\lambda \in \ker(\mathcal{L})$ can exist; consequently, $\ker(\mathcal{L}) = \{0\}$.

If $v_\infty(h) < 0$, then $\ker(\mathcal{L}) = \{0\}$ by Remark 5.1 (i). So $I_h = \{0\}$. $\square$

We distinguish the trivial and nontrivial cases of I_h using the following notion of type.

Definition 5.10. The *type* of I_h is defined as 0 if $I_h = \{0\}$ (i.e., $\ker(\mathcal{L}) = \{0\}$) and otherwise as an element $\lambda \in \ker(\mathcal{L})^\times$.

The type of I_h is unique up to a multiplicative constant in $C^\times$, because $\dim_C \ker(\mathcal{L}) \leq 1$. It can be found by recognizing logarithmic derivatives in F .

Example 5.11. Let $h = 0$. Then $m = 0$, $a_m = 0$ and $b_m = 1$ by Convention 4.4. Thus, $\mathcal{L} = ' by (8). Consequently, $\ker(\mathcal{L}) = C$. The type of I_0 can be taken as 1.$

5.3. Echelon sequences and induced complements

The goal of this subsection is to construct an echelon sequence of I_h and use it to obtain a complement of $\text{im}(\mathcal{P}_h)$ in $F[t]$. Let U_h be the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and set $I_h := \text{im}(\mathcal{P}_h) \cap U_h$ as before. Assume further that $I_h \neq \{0\}$ and its type is a fixed element $\lambda \in \ker(\mathcal{L})^\times$. We proceed in the following four steps:

1. Construct a C -basis $B_1 = \{p_i\}_{i \in \mathbb{N}_0}$ of $\mathcal{P}_h^{-1}(I_h)$ in Lemma 5.14.
2. Describe a C -basis B_2 of I_h in Proposition 5.15 by applying $\mathcal{P}_h$ to B_1 and omitting $\mathcal{P}_h(p_0)$ when $h \in F$.
3. Convert B_2 into an echelon sequence of I_h in Corollaries 5.17 and 5.18.
4. Construct a complement of $\text{im}(\mathcal{P}_h)$ in $F[t]$ by Lemma 2.9.

In the rest of this subsection, every basis is a C -basis. Thus, we omit the prefix “ C -” for brevity.

5.3.1. Standard basis of $\mathcal{P}_h^{-1}(I_h)$

Recall the notation given in Convention 4.4. We have

$$m = \text{Deg}(h), \quad \text{num}(h) = \sum_{i=0}^m a_i t^i \quad \text{and} \quad \text{den}(h) = \sum_{i=0}^m b_i t^i.$$

It follows from Proposition 5.9 that $v_\infty(h) \geq 0$. By Remark 5.1 (ii), $b_m = 1$ and $\mathcal{L} = \mathcal{R}_{a_m}$.

In view of Lemma 5.8 and $\dim_C(\ker(\mathcal{L})) = 1$, all elements of $\mathcal{P}_h^{-1}(I_h)^\times$ have leading coefficient $c\lambda$ for some $c \in C^\times$. This observation motivates us to compute the leading term of $\mathcal{P}_h(\lambda t^i)$ for all $i \in \mathbb{N}_0$. By Definition 3.5 and $\mathcal{L}(\lambda) = 0$, we have

$$\mathcal{P}_h(\lambda t^i) \equiv (i\lambda t' + b_{m-1}\lambda' + a_{m-1}\lambda) \cdot t^{m+i-1} \pmod{F[t]_{< m+i-1}} \quad (11)$$

if $m + i > 0$. To express the coefficient $i\lambda t' + b_{m-1}\lambda' + a_{m-1}\lambda$ of t^{m+i-1} in terms of R -pairs, we introduce the following definition.

Definition 5.12. Let Φ_{a_m} be the complete reduction for $(F, \mathcal{R}_{a_m})$ given in Hypothesis 4.1. A *first R -pair associated to $(F[t], \mathcal{P}_h)$* is an R -pair of $\lambda t'$ with respect to Φ_{a_m} , while a *second R -pair* is an R -pair of $b_{m-1}\lambda' + a_{m-1}\lambda$ with respect to Φ_{a_m} .

An element may admit R -pairs with different first components. For the rest of this section, we fix the first and second associated R -pairs and denote them by $(\tilde{\sigma}, \sigma)$ and $(\tilde{\tau}, \tau)$, respectively. Note that $(\tilde{\tau}, \tau) := (0, 0)$ if $m = 0$.

Remark 5.13. The entry σ in the first associated R-pair $(\tilde{\sigma}, \sigma)$ is nonzero by Lemma 5.2.

With

$$\lambda t' = \mathcal{R}_{a_m}(\tilde{\sigma}) + \sigma \quad \text{and} \quad b_{m-1}\lambda' + a_{m-1}\lambda = \mathcal{R}_{a_m}(\tilde{\tau}) + \tau,$$

we rewrite (11) as

$$\mathcal{P}_h(\lambda t^i) \equiv \mathcal{R}_{a_m}(i\tilde{\sigma} + \tilde{\tau}) \cdot t^{m+i-1} + (i\sigma + \tau) \cdot t^{m+i-1} \pmod{F[t]_{<m+i-1}}. \quad (12)$$

To construct a basis for $\mathcal{P}_h^{-1}(I_h)$, we first define a sequence $\{p_i\}_{i \in \mathbb{N}_0}$ in $F[t]$ such that $\deg(p_i) = i$, $\text{lc}(p_i) = \lambda$ and $p_i \in \mathcal{P}_h^{-1}(I_h)$.

For $i = 0$, we let $p_0 = \lambda$. Then $\mathcal{P}_h(p_0) \in F[t]_{<m}$ by (9). Since $F[t]_{<m} \subset U_h$, we have $\mathcal{P}_h(p_0) \in I_h$ and, consequently, $p_0 \in \mathcal{P}_h^{-1}(I_h)$.

Assume that $i > 0$. It follows from (12) and $\mathcal{L} = \mathcal{R}_{a_m}$ that

$$\mathcal{P}_h(\lambda t^i) \equiv \mathcal{L}(i\tilde{\sigma} + \tilde{\tau}) \cdot t^{m+i-1} + (i\sigma + \tau) \cdot t^{m+i-1} \pmod{F[t]_{<m+i-1}},$$

which, together with (9), implies that there exists a polynomial $g_i \in F[t]_{<m+i-1}$ such that

$$\mathcal{P}_h(\lambda t^i) = \mathcal{P}_h((i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1}) + (i\sigma + \tau) \cdot t^{m+i-1} + g_i.$$

Let (q_i, r_i) be the auxiliary pair of g_i obtained from Algorithm A.1. Then

$$\deg(q_i) < i - 1 \quad \text{and} \quad \deg(r_i) < m + i - 1$$

by Proposition 5.5. Moreover,

$$\mathcal{P}_h(\lambda t^i) = \mathcal{P}_h((i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1}) + (i\sigma + \tau) \cdot t^{m+i-1} + \mathcal{P}_h(q_i) + r_i.$$

Moving the images under $\mathcal{P}_h$ in the above equality to the left-hand side, we arrive at

$$\mathcal{P}_h(\lambda t^i - (i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1} - q_i) = (i\sigma + \tau) \cdot t^{m+i-1} + r_i.$$

Set

$$p_i := \lambda t^i - (i\tilde{\sigma} + \tilde{\tau}) \cdot t^{i-1} - q_i. \quad (13)$$

Then $\deg(p_i) = i$ and

$$\mathcal{P}_h(p_i) = (i\sigma + \tau) \cdot t^{m+i-1} + r_i. \quad (14)$$

Since $\sigma, \tau \in \text{im}(\Phi_{a_m})$ and $r_i \in U_h$, we see that $\mathcal{P}_h(p_i) \in U_h$, that is, $p_i \in \mathcal{P}_h^{-1}(I_h)$ for all $i \in \mathbb{N}$.

Lemma 5.14. Let $p_0 = \lambda$ and p_i be defined by (13) for all $i \in \mathbb{N}$. Then $\{p_i\}_{i \in \mathbb{N}_0}$ is a basis of $\mathcal{P}_h^{-1}(I_h)$.

Proof. We have seen that $\{p_i\}_{i \in \mathbb{N}_0} \subset \mathcal{P}_h^{-1}(I_h)$. Since $\deg(p_i) = i$ for all $i \in \mathbb{N}_0$, the set $\{p_i\}_{i \in \mathbb{N}_0}$ is C -linearly independent. Let $f \in F[t]^\times$ satisfy $\mathcal{P}_h(f) \in U_h$. By Lemma 5.8, $\text{lc}(f) \in \ker(\mathcal{L})$. Then $\text{lc}(f) = c\lambda$ for some $c \in C^\times$ by $\lambda \in \ker(\mathcal{L})^\times$ and $\dim_C(\ker(\mathcal{L})) = 1$.

Let $d = \deg(f)$. Then

$$f - cp_d \in \mathcal{P}_h^{-1}(I_h)$$

with $\deg(f - cp_d) < d$. A straightforward induction on d then shows that f is a C -linear combination of $p_d, \dots, p_0$. $\square$

We call $\{p_i\}_{i \in \mathbb{N}_0}$ the *standard basis* of $\mathcal{P}_h^{-1}(I_h)$, where $p_0 = \lambda$ and p_i is given in (13) for $i \in \mathbb{N}$. This basis is unique, because the type of I_h and the two associated R-pairs are all fixed, and every auxiliary pair (q_i, r_i) is computed by Algorithm A.1.

Let us summarize the notation introduced so far in the following table.

Table 1
Notational summary in Section 5.3.1.

	Type	1st R-pair	2nd R-pair	Standard basis
Expression	λ	$(\tilde{\sigma}, \sigma)$	$(\tilde{\tau}, \tau)$	$p_0 = \lambda, p_i$ in (13), $i \in \mathbb{N}$
Property	$\lambda \in \ker(\mathcal{L})^\times$	$\lambda t' = \mathcal{R}_{a_m}(\tilde{\sigma}) + \sigma$	$b_{m-1}\lambda' + a_{m-1}\lambda$ $= \mathcal{R}_{a_m}(\tilde{\tau}) + \tau$	A basis of $\mathcal{P}_h^{-1}(I_h)$

5.3.2. Basis of I_h

The companion operator $\mathcal{P}_h$ enables us to transform the standard basis above into a basis of I_h .

Proposition 5.15. *With the notation given in Table 1, we further let r_i be given in (14).*

(i) *If $h \in F$, then $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}}$ is a basis of I_h , and for all $i \in \mathbb{N}$,*

$$\mathcal{P}_h(p_i) = i\sigma t^{i-1} + r_i \quad \text{and} \quad \deg(\mathcal{P}_h(p_i)) = i - 1.$$

(ii) *If $h \in F(t) \setminus F$, then $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}_0}$ is a basis of I_h , $\mathcal{P}_h(p_0) \in F[t]_{<m}^\times$, and for all $i \in \mathbb{N}$,*

$$\mathcal{P}_h(p_i) = (i\sigma + \tau)t^{m+i-1} + r_i.$$

Proof. (i) Since $h \in F$, the restriction of $\mathcal{P}_h$ to F is equal to $\mathcal{L}$. Then $\mathcal{P}_h(p_0) = 0$ by $p_0 = \lambda$. Hence, $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}}$ spans I_h over C by Lemma 5.14. Moreover, $h \in F$ implies that $m = 0$ and $(\tilde{\tau}, \tau) = (0, 0)$. It follows from (14) that $\mathcal{P}_h(p_i) = i\sigma t^{i-1} + r_i$. By Remark 5.13 and $\deg(r_i) < i - 1$, $\mathcal{P}_h(p_i)$ has degree $i - 1$. Thus, $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}}$ is a basis of I_h .

(ii) Since $h \in F(t) \setminus F$, the companion operator $\mathcal{P}_h$ is injective by Lemma 4.3, which, together with Lemma 5.14, implies that $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}_0}$ is a basis of I_h . Note that $\mathcal{P}_h(p_0) = \mathcal{P}_h(\lambda)$, which is a nonzero polynomial of degree less than m by $\lambda \in \ker(\mathcal{L})$ and (9). The rest follows by (14). $\square$

The following example rediscovers the complete reduction in Example 1.2.

Example 5.16. Let $F = C$ and $t = x$ with $x' = 1$. Then $F(t) = C(x)$. Set $h = 0$. Then the auxiliary subspace U_0 is equal to $C[x]$ by Example 5.4. The calculations carried out for constructing the standard basis of $\mathcal{P}_0^{-1}(I_0)$ lead to the following table:

Type λ	1st R-pair $(\tilde{\sigma}, \sigma)$	2nd R-pair $(\tilde{\tau}, \tau)$	$p_i, i \in \mathbb{N}_0$	$\mathcal{P}_0(p_i), i \in \mathbb{N}$
1	$(0, 1)$	$(0, 0)$	x^i	ix^{i-1}

Then $I_0 = C[x]$ by Proposition 5.15 (i). Hence, $C[x] = \text{im}(\mathcal{P}_0)$. Consequently, $\{0\}$ is the only complement of $\text{im}(\mathcal{P}_0)$ in $C[x]$. By Proposition 3.6, $C(x) = C(x)' \oplus S_x$.

5.3.3. Echelon sequences

We now construct echelon sequences of I_h separately for $h \in F$ and $h \in F(t) \setminus F$. Echelon sequences are defined in Definition 2.10.

Let Θ be an effective basis of F (see Definition 2.12). Then

$$\Omega := \{\theta t^i \mid \theta \in \Theta, i \in \mathbb{N}_0\} \quad (15)$$

is an effective basis of $F[t]$ by Remark 2.7 in (Du et al., 2025).

Recall that $(\tilde{\sigma}, \sigma)$ and $(\tilde{\tau}, \tau)$ are the first and second R-pairs associated to $(F[t], \mathcal{P}_h)$, respectively. In the rest of this section, we fix an element $\theta_\sigma \in \Theta$ with $\theta_\sigma^*(\sigma) \neq 0$. By Remark 5.13, such an element exists. It will be used to describe most of the pivots in an echelon sequence.

The next corollary is immediate from Proposition 5.15 (i).

Corollary 5.17. With the notation in Table 1, we further let $h \in F$. Then

$$\left\{ \left(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{i-1} \right) \right\}_{i \in \mathbb{N}}$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω .

When $h \in F(t) \setminus F$, constructing an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω requires a detailed case analysis because the elements of $\{\mathcal{P}_h(p_i)\}_{i \in \mathbb{N}_0}$ in Proposition 5.15 (ii) do not necessarily have distinct degrees. Such an analysis is carried out in (Bostan et al., 2013, Section 4.1) for the special case in which $F = \mathbb{C}$, $t = x$ and $' = d/dx$.

Corollary 5.18. With the notation given in Table 1, we further let $h \in F(t) \setminus F$ satisfy $\text{Deg}(h) = m$. Assume that d_0 and l_0 are the degree and leading coefficient of $\mathcal{P}_h(p_0)$, respectively, and set θ_0 to be an element of Θ such that $\theta_0^*(l_0) \neq 0$.

(i) If $\theta_\sigma^*(i\sigma + \tau) \neq 0$ for all $i \in \mathbb{N}$, then

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0} \right), \left(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^m \right), \left(p_2, \mathcal{P}_h(p_2), \theta_\sigma t^{m+1} \right), \dots, \left(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{m+i-1} \right), \dots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω , illustrated below:

preimages	basis elements	degrees of basis elements	pivots
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m + i - 1$	$\theta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\theta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 with $d_0 < m$	$\theta_0 t^{d_0}$

where the length of each line segment corresponds to the degree of the corresponding basis element; the same convention is used in the next two pictures.

(ii) If $\theta_\sigma^*(j\sigma + \tau) = 0$ for some $j \in \mathbb{N}$ but $j\sigma + \tau \neq 0$, then there exists $\theta \in \Theta$ with $\text{lc}(\mathcal{P}_h(p_j)) \notin \ker(\theta^*)$ such that

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0} \right), \left(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^m \right), \dots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_\sigma t^{m+j-2} \right), \\ \left(p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1} \right), \left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \theta_\sigma t^{m+j} \right), \dots, \left(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{m+i-1} \right), \dots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω , illustrated below:

preimages	basis elements	degrees of basis elements	pivots
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m + i - 1$	$\theta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$m + j$	$\theta_\sigma t^{m+j}$
p_j	$\mathcal{P}_h(p_j)$	$m + j - 1$	θt^{m+j-1}
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$m + j - 2$	$\theta_\sigma t^{m+j-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\theta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 with $d_0 < m$	$\theta_0 t^{d_0}$

(iii) If $j\sigma + \tau = 0$ for some $j \in \mathbb{N}$, then there exist $q \in F[t]^\times$, $d \in \mathbb{N}_0$, and $\theta \in \Theta$ with $\text{lc}(\mathcal{P}_h(q)) \notin \ker(\theta^*)$ such that

$$\left(q, \mathcal{P}_h(q), \theta t^d \right), \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0} \right), \left(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^m \right), \dots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_\sigma t^{m+j-2} \right), \\ \left(p_{j+1}, \mathcal{P}_h(p_{j+1}), \theta_\sigma t^{m+j} \right), \dots, \left(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{m+i-1} \right), \dots$$

is an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω , illustrated below:

preimages	basis elements	degrees of basis elements	pivots
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_i	$\mathcal{P}_h(p_i)$	$m+i-1$	$\theta_\sigma t^{m+i-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{j+1}	$\mathcal{P}_h(p_{j+1})$	$m+j$	$\theta_\sigma t^{m+j}$
p_{j-1}	$\mathcal{P}_h(p_{j-1})$	$m+j-2$	$\theta_\sigma t^{m+j-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_1	$\mathcal{P}_h(p_1)$	m	$\theta_\sigma t^m$
p_0	$\mathcal{P}_h(p_0)$	d_0 with $d_0 < m$	$\theta_0 t^{d_0}$
q	$\mathcal{P}_h(q)$	d with $d < m+j-1$	θt^d

Proof. (i) The conclusion is immediate from Proposition 5.15 (ii).

(ii) Since $j\sigma + \tau \neq 0$, we see that $\mathcal{P}_h(p_j)$ has degree $m+j-1$ by Proposition 5.15 (ii). Note that j is the unique integer such that $\theta_\sigma^*(j\sigma + \tau) = 0$. So $\theta_\sigma^*(i\sigma + \tau) \neq 0$ for all $i \in \mathbb{N}$ with $i \neq j$. Hence, $\mathcal{P}_h(p_i)$ has degree $m+i-1$ for all $i \in \mathbb{N}$ with $i \neq j$. Let θ be an element of Θ such that $\text{lc}(\mathcal{P}_h(p_j)) \notin \ker(\theta^*)$. This proves (ii).

(iii) Since $j\sigma + \tau = 0$, the degree of $\mathcal{P}_h(p_j)$ is less than $m+j-1$ by Proposition 5.15 (ii). Let U be the subspace spanned by $\mathcal{P}_h(p_0), \mathcal{P}_h(p_1), \dots, \mathcal{P}_h(p_{j-1})$ over C . Then

$$\left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0} \right), \left(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^m \right), \dots, \left(p_{j-1}, \mathcal{P}_h(p_{j-1}), \theta_\sigma t^{m+j-2} \right)$$

is an echelon sequence of U with respect to $(F[t], \mathcal{P}_h)$ and Ω .

Set $\omega_0 := \theta_0 t^{d_0}$, $\omega_1 := \theta_\sigma t^m$, $\dots$, $\omega_{j-1} := \theta_\sigma t^{m+j-2}$. It follows from Lemma 2.13 that there exists a polynomial $r \in F[t]$ such that $\mathcal{P}_h(p_j) - \mathcal{P}_h(r) \in \bigcap_{k=0}^{j-1} \ker(\omega_k^*)$ and $\mathcal{P}_h(r) \in U$.

Set $q := p_j - r$. Then $\mathcal{P}_h(q) \in I_h \cap \left(\bigcap_{k=0}^{j-1} \ker(\omega_k^*) \right)$. Moreover, since $\mathcal{P}_h(r)$ lies in U , replacing $\mathcal{P}_h(p_j)$ by $\mathcal{P}_h(q) = \mathcal{P}_h(p_j) - \mathcal{P}_h(r)$ preserves the basis property. Thus

$$\mathcal{P}_h(q), \mathcal{P}_h(p_0), \dots, \mathcal{P}_h(p_{j-1}), \mathcal{P}_h(p_{j+1}), \mathcal{P}_h(p_{j+2}), \dots$$

form a basis of I_h . Let d be the degree of $\mathcal{P}_h(q)$, and choose $\theta \in \Theta$ such that $\text{lc}(\mathcal{P}_h(q)) \notin \ker(\theta^*)$. This proves (iii). $\square$

5.3.4. Induced complements

Let $E := \{(f_i, \mathcal{P}_h(f_i), \omega_i)\}_{i \in \mathbb{N}}$ be an echelon sequence of I_h obtained from Corollary 5.17 or 5.18, reindexed by $\mathbb{N}$ if necessary. It follows from Lemma 2.9 that the subspace

$$U_h \cap \left(\bigcap_{i \in \mathbb{N}} \ker(\omega_i^*) \right) \quad (16)$$

is a complement of $\text{im}(\mathcal{P}_h)$ in $F[t]$. We call it the *complement of $\text{im}(\mathcal{P}_h)$ induced by E* .

Any complement of $\text{im}(\mathcal{P}_h)$ in $F[t]$ is finite-dimensional over C if $F = C$ by Lemma 8 in (Bostan et al., 2013) and Example 5.16. However, such a complement is infinite-dimensional if $\dim_C(F) = \infty$.

Example 5.19. Let $F = C(x)$ and $t = \log(x)$, with d/dx as the derivation on $F(t)$. Let

$$h = \frac{2x^2 - 2t}{xt^2 + x},$$

which is t -normalized by direct verification.

We construct an echelon sequence of I_h with respect to $(F[t], \mathcal{P}_h)$ and Ω in (15). By Convention 4.4, we have the following table:

a	b	m	a_m	b_m	a_{m-1}	b_{m-1}
$-2x^{-1}t + 2x$	$t^2 + 1$	2	0	1	$-2x^{-1}$	0

Since $a_m = 0$, we may take the Hermite-Ostrogradsky reduction to be the complete reduction Φ_0 for $(F, ')$ in Hypothesis 4.1. The auxiliary subspace U_h associated to $(F[t], \mathcal{P}_h)$ is equal to

$$F[t]_{<2} + \sum_{i \in \mathbb{N}_0} S_x \cdot t^{i+2} \quad (17)$$

by Definition 5.3 and $\text{im}(\Phi_0) = S_x$.

Since $a_2 = 0$ and $b_2 = 1$, the operator $\mathcal{L}$ in (8) is the derivation d/dx . Then we may take the type λ of I_h to be 1. By Algorithm A.1, the first associated R-pair is $(\tilde{\sigma}, \sigma) = (0, x^{-1})$ and the second R-pair is $(\tilde{\tau}, \tau) = (0, -2x^{-1})$. Since $\sigma = x^{-1}$, we choose θ_σ to be x^{-1} . Corollary 5.18 (iii) is needed to construct an echelon sequence E , because $2\sigma + \tau = 0$.

We first compute p_0 and p_1 in the standard basis $\{p_i\}_{i \in \mathbb{N}_0}$ of $\mathcal{P}_h^{-1}(I_h)$. Since the type λ is equal to 1, we have $p_0 = 1$, and then $\mathcal{P}_h(p_0) = -2x^{-1}t + 2x$. Hence, $d_0 = 1$. Choose $\theta_0 = x^{-1}$. Then the second term of E is

$$(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}) = (1, -2x^{-1}t + 2x, x^{-1}t). \quad (18)$$

Note that $\theta_0 = \theta_\sigma$ is just a coincidence. By (13) and $\tilde{\sigma} = \tilde{\tau} = 0$, we see that $p_1 = t$. Thus, the third term of E is

$$(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^2) = (t, -x^{-1}t^2 + 2xt + x^{-1}, x^{-1}t^2). \quad (19)$$

Since $2\sigma + \tau = 0$, we compute the first term of E using p_2 . By (13) and Algorithm A.1, we find $p_2 = t^2 - x^2$. Then $\mathcal{P}_h(p_2) = (2x + 2x^{-1})t - 2x^3 - 2x$. Eliminating $x^{-1}t$ from $\mathcal{P}_h(p_2)$ using $\mathcal{P}_h(p_0)$ yields $(q, \mathcal{P}_h(q)) = (t^2 - x^2 + 1, 2xt - 2x^3)$. Choose $\theta = x$. Then the first term is

$$(q, \mathcal{P}_h(q), \theta t) = (t^2 - x^2 + 1, 2xt - 2x^3, x t). \quad (20)$$

For every $k > 3$, the k th term of E can be computed using (13), (14), and Algorithm A.1. Its pivot is $x^{-1}t^k$ by Corollary 5.18 (iii). We summarize the results in the following table:

λ	$(\tilde{\sigma}, \sigma)$	$(\tilde{\tau}, \tau)$	θ_σ	j	$(q, \mathcal{P}_h(q), \theta t)$	$(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0})$	$(p_1, \mathcal{P}_h(p_1), \theta_\sigma t^2)$
1	$(0, x^{-1})$	$(0, -2x^{-1})$	x^{-1}	2	r.h.s. of (20)	r.h.s. of (18)	r.h.s. of (19)

where j denotes the unique positive integer satisfying $j\sigma + \tau = 0$.

By (16) and (17), the complement W of $\text{im}(\mathcal{P}_h)$ induced by E is equal to

$$(C(x)[t]_{<2} + \sum_{i \in \mathbb{N}_0} S_x \cdot t^{i+2}) \cap \left(\bigcap_{i \in \mathbb{N}} \ker(\omega_i^*) \right),$$

where $\omega_1 = xt$, $\omega_2 = x^{-1}t$, $\omega_3 = x^{-1}t^2$ as listed in the table above, and $\omega_k = x^{-1}t^k$ for $k > 3$ by the preceding description of the pivots. Any element $r \in W$ can be written in the form

$$r = r_0 + r_1 t + r_2 t^2 + r_3 t^3 + r_4 t^4 + \cdots + r_k t^k$$

with $r_i \in C(x)$ satisfying the following conditions:

- r_0 is an arbitrary element of $C(x)$;
- in the irreducible partial fraction decomposition of r_1 , the polynomial part has no term of the form cx , and the proper part contains no component of the form c/x for any $c \in C^\times$;
- r_3 is an arbitrary element of S_x ;
- the coefficients r_2 and r_i with $4 \leq i \leq k$ all belong to S_x and have no pole at $x = 0$.

We have made three choices of pivots in the above example. Together with (13) and Algorithm A.1, these choices uniquely determine an echelon sequence of I_h . To systematically study such constructions, we introduce the notion of initial sequences in Definition A.2 of the appendix. Every initial sequence is finite and contains all the information required to construct a unique echelon sequence. For instance, the final table in the above example encodes an initial sequence. Moreover, Algorithm A.4 computes initial sequences, and Algorithm A.5 generates arbitrarily many terms of the echelon sequence determined by a given initial sequence.

In view of Remark 1.6, we are ready to present the main result of this section.

Theorem 5.20. *Assume that Hypothesis 4.1 holds and t is primitive. Then there exists an algorithm that, for every t -normalized element $h \in F(t)$, constructs a complete reduction $\tilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.*

Proof. Recall that U_h is the auxiliary subspace associated to $(F[t], \mathcal{P}_h)$, and that λ is the type of I_h . If $\lambda = 0$, then $I_h = \{0\}$ by Definition 5.10. Hence, Proposition 5.5 gives $F[t] = \text{im}(\mathcal{P}_h) \oplus U_h$, and we set $W := U_h$. Otherwise, let E be the echelon sequence of I_h obtained from Corollary 5.17 or Corollary 5.18. Set W to be the complement of $\text{im}(\mathcal{P}_h)$ induced by E . Then $F[t] = \text{im}(\mathcal{P}_h) \oplus W$.

It follows from Proposition 3.6 that

$$F(t) = \text{im}(\mathcal{R}_h) \oplus (b^{-1} \cdot W \oplus S_{t,h}), \quad (21)$$

where $b = \text{den}(h)$ by Convention 4.4, and $S_{t,h}$ is given in (4). Therefore, the projection from $F(t)$ onto $(b^{-1} \cdot W \oplus S_{t,h})$ with respect to (21) gives a complete reduction $\tilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.

It remains to describe an algorithm for computing R-pairs with respect to $\tilde{\Phi}_h$. Let $\tilde{\Psi}_h$ be the projection from $F[t]$ onto W with respect to $F[t] = \text{im}(\mathcal{P}_h) \oplus W$. By Remark 3.7, it suffices to develop an algorithm for computing R-pairs with respect to $\tilde{\Psi}_h$.

Take any $p \in F[t]$. An auxiliary pair (q, r) of p can be found by Algorithm A.1. If $\lambda = 0$, then (q, r) is an R-pair with respect to $\tilde{\Psi}_h$. Otherwise, one can compute $\tilde{q} \in F[t]$ and $\tilde{r} \in I_h$ such that $\tilde{r} = \mathcal{P}_h(\tilde{q})$ and $r - \tilde{r} \in \bigcap_{i \in \mathbb{N}} \ker(\omega_i^*)$ by Lemma 2.13, where ω_i denotes the pivot of the i th term of E . On the other hand, $r - \tilde{r} \in U_h$ by $r, \tilde{r} \in U_h$. Thus, $r - \tilde{r} \in W$ by (16). Consequently, $(q + \tilde{q}, r - \tilde{r})$ is an R-pair of p with respect to $\tilde{\Psi}_h$. $\square$

The next corollary relates the complete reduction $\tilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$ from the preceding theorem to the complete reduction Φ_h for $(F, \mathcal{R}_h)$ specified in Hypothesis 4.1 whenever $h \in F$. It is useful for elementary integration in Section 8.

Corollary 5.21. *Let $h \in F$, Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1, and $\tilde{\Phi}_h$ be the complete reduction for $(F(t), \mathcal{R}_h)$ in Theorem 5.20. The following two assertions hold.*

- For all $f \in F$, $\tilde{\Phi}_h(f) = \Phi_h(f)$ if $I_h = \{0\}$, and, otherwise, $\tilde{\Phi}_h(f) = \Phi_h(f) + c\Phi_h(\lambda t')$ for some constant $c \in C$, where λ is the type of I_h .
- $\text{im}(\tilde{\Phi}_0) \subset \text{im}(\Phi_0) \oplus (t \cdot F[t]) \oplus S_t$.

Proof. (i) Since $h \in F$, we see that $b = 1$ by Convention 4.4 (iv). Then $S_{t,h}$ in (21) is equal to S_t . Hence, equation (21) becomes

$$F(t) = \text{im}(\mathcal{R}_h) \oplus (W \oplus S_t). \quad (22)$$

Moreover, $\tilde{\Phi}_h$ is the projection from $F(t)$ onto $W \oplus S_t$ with respect to this direct sum.

Let $f \in F$. The assertion is immediate for $f = 0$. Assume that $f \neq 0$. By Corollary 5.7,

$$f = \mathcal{R}_h(\alpha) + \Phi_h(f) \quad (23)$$

for some $\alpha \in F$, and $(\alpha, \Phi_h(f))$ is an auxiliary pair of f with respect to $(F[t], \mathcal{P}_h)$.

If $I_h = \{0\}$, then W in (22) is equal to U_h . Since $\mathcal{R}_h(\alpha) \in \text{im}(\mathcal{R}_h)$ and $\Phi_h(f) \in U_h$, we conclude by (23) that $\tilde{\Phi}_h(f) = \Phi_h(f)$.

Assume that $I_h \neq \{0\}$. Then W in (22) is the complement induced by the echelon sequence E of I_h given by Corollary 5.17. It follows from $\Phi_h(f) \in U_h$ and (23) that $\tilde{\Phi}_h(f)$ is the projection of $\Phi_h(f)$ onto W with respect to $U_h = I_h \oplus W$. Let $(\tilde{\sigma}, \sigma)$ be the first R-pair associated to $(F[t], \mathcal{P}_h)$. Then $\sigma = \Phi_h(\lambda t')$ by Definition 5.12. Note that the first term of E is

$$(\lambda t - \tilde{\sigma}, \Phi_h(\lambda t'), \theta_\sigma).$$

Set $c := -\theta_\sigma^* (\Phi_h(\lambda t'))^{-1} \cdot \theta_\sigma^* (\Phi_h(f))$. It follows that $\Phi_h(f) + c\Phi_h(\lambda t')$ belongs to W , because all other pivots in E have positive degree in t . Therefore, $\tilde{\Phi}_h(f) = \Phi_h(f) + c\Phi_h(\lambda t')$.

(ii) By Definition 5.3, the auxiliary subspace associated to $(F[t], \mathcal{P}_0)$ is

$$U_0 = \sum_{i \in \mathbb{N}_0} \text{im}(\Phi_0) \cdot t^i = \text{im}(\Phi_0) + \sum_{i \in \mathbb{N}} \text{im}(\Phi_0) \cdot t^i,$$

which is contained in $\text{im}(\Phi_0) + (t \cdot F[t])$. Since $W \subset U_0$, we have $W \subset \text{im}(\Phi_0) + (t \cdot F[t])$. Thus, $\text{im}(\tilde{\Phi}_0) \subset \text{im}(\Phi_0) + (t \cdot F[t]) + S_t$. The sum of the three subspaces is direct. $\square$

6. The hyperexponential case

This section is organized similarly to Section 5, with two substantive differences. First, the definition of auxiliary subspaces becomes more involved due to the presence of negative powers. Second, the construction of echelon sequences turns out to be considerably simpler, since the intersection of $\text{im}(\mathcal{P}_h)$ with the auxiliary subspace associated to $(F(t), \mathcal{P}_h)$ has dimension at most two over C (see Proposition 6.13). The results in this section generalize those in (Gao, 2024, Chapter 4) for the exponential case and (Chen et al., 2025a, Section 4.2) for the rationally hyperexponential case.

We keep the assumptions and notation in Convention 4.4. Then t is hyperexponential and $F(t) = F[t, t^{-1}]$. The tail coefficients a_0 and b_0 will play an important role in reducing negative powers of t .

To describe the head and tail terms of an element in $\text{im}(\mathcal{P}_h)$, we define two C -linear operators as follows. For every $d \in \mathbb{Z}$, let

$$\begin{aligned} \mathcal{H}_d : F &\rightarrow F & \mathcal{T}_d : F &\rightarrow F \\ \alpha &\mapsto b_m \alpha' + \left(a_m + db_m \frac{t'}{t} \right) \alpha & \text{and} & \quad \alpha \mapsto b_0 \alpha' + \left(a_0 + db_0 \frac{t'}{t} \right) \alpha. \end{aligned} \quad (24)$$

For every $f \in F[t, t^{-1}]^\times$ with $k = \text{hdeg}(f)$, $f_k = \text{hc}(f)$, $l = \text{tdeg}(f)$ and $f_l = \text{tc}(f)$, we have

$$\mathcal{P}_h(f) = \mathcal{H}_k(f_k)t^{m+k} + \dots + \mathcal{T}_l(f_l)t^l \quad (25)$$

by Lemma 2.2 (ii). It follows that $\mathcal{P}_h(f^+) \in F[t]$ and $\mathcal{P}_h(f^-) \in F[t]_{<m} + t^{-1}F[t^{-1}]$.

Remark 6.1. By Convention 4.4 and (24), the following assertions hold for all $k, l \in \mathbb{Z}$.

- (i) If $v_\infty(h) < 0$, then $a_m \neq 0$, $b_m = 0$ and $\ker(\mathcal{H}_k) = \{0\}$.
- (ii) If $v_\infty(h) \geq 0$, then $b_m = 1$ and $\mathcal{H}_k = \mathcal{R}_{\lambda_k}$, where $\lambda_k = a_m + k \frac{t'}{t}$.
- (iii) If $v_t(h) < 0$, then $a_0 \neq 0$, $b_0 = 0$, and $\ker(\mathcal{T}_l) = \{0\}$.
- (iv) If $v_t(h) \geq 0$, then $b_0 \neq 0$ and $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$, where $\mu_l = \frac{a_0}{b_0} + l \frac{t'}{t}$.

The λ_k 's and μ_l 's defined above will be used throughout the rest of this section.

Remark 6.2. There exists at most one integer k (resp. one integer l) such that $\ker(\mathcal{H}_k) \neq \{0\}$ (resp. $\ker(\mathcal{T}_l) \neq \{0\}$) by Lemma 6.2.2 in (Bronstein, 2005).

In this section, we mainly establish results concerning tail degrees. The head-degree case is analogous and is in fact simpler, since $b_m \in \{0, 1\}$.

6.1. Auxiliary subspaces

As in Section 5.1, we aim to reduce all coefficients of elements in $F[t, t^{-1}]$ modulo $\text{im}(\mathcal{P}_h)$.

Let $f \in F[t, t^{-1}]$ with $k = \text{hdeg}(f)$, $f_k = \text{hc}(f)$, $l = \text{tdeg}(f)$ and $f_l = \text{tc}(f)$. Assume further that $k \geq m$ and $l < 0$. By arguments analogous to those at the beginning of Section 5.1, we obtain the following four congruences.

$$f^+ \equiv \begin{cases} \mathcal{P}_h(a_m^{-1} f_k t^{k-m}) \mod F[t]_{<k} & \text{if } v_\infty(h) < 0, \\ \mathcal{P}_h(\alpha t^{k-m}) + \beta t^k \mod F[t]_{<k} & \text{if } v_\infty(h) \geq 0, \end{cases} \quad (26)$$

where $(\alpha, \beta) \in F^2$ is an R-pair of f_k with respect to $\Phi_{\lambda_{k-m}}$ given in Hypothesis 4.1, and

$$f^- \equiv \begin{cases} \mathcal{P}_h(a_0^{-1} f_l t^l) \mod F[t]_{<m} + (t^{-1} \cdot F[t^{-1}])_{>l} & \text{if } v_t(h) < 0, \\ \mathcal{P}_h(\alpha t^l) + (b_0 \beta) t^l \mod F[t]_{<m} + (t^{-1} \cdot F[t^{-1}])_{>l} & \text{if } v_t(h) \geq 0, \end{cases} \quad (27)$$

where $(\alpha, \beta) \in F^2$ is an R-pair of $b_0^{-1} f_l$ with respect to Φ_{μ_l} in Hypothesis 4.1. Note that $v_t(h) < 0$ implies $a_0 \neq 0$ by Remark 6.1 (iii) and $v_t(h) \geq 0$ implies $b_0 \neq 0$ by Remark 6.1 (iv).

The two congruences in (26) can be verified by the same reasoning used to establish (10). The first congruence in (27) holds by (25) and Remark 6.1 (iii). To show the second congruence in (27), we note that $b_0^{-1} f_l = \mathcal{R}_{\mu_l}(\alpha) + \beta$, which, together with $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$ in Remark 6.1 (iv), implies that $f_l = \mathcal{T}_l(\alpha) + b_0 \beta$. Thus, $f^- \equiv \mathcal{T}_l(\alpha) t^l + (b_0 \beta) t^l \mod F[t]_{<m} + (t^{-1} \cdot F[t^{-1}])_{>l}$. The congruence follows from (25).

The preceding discussion motivates the following notion of an auxiliary subspace.

Definition 6.3. The auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$ is defined as

$$V_h = V_h^+ + F[t]_{<m} + V_h^-,$$

where

$$V_h^+ = \begin{cases} \{0\} & \text{if } v_\infty(h) < 0, \\ \sum_{i \geq m} \text{im}(\Phi_{\lambda_{i-m}}) \cdot t^i & \text{if } v_\infty(h) \geq 0, \end{cases} \quad \text{and } V_h^- = \begin{cases} \{0\} & \text{if } v_t(h) < 0, \\ \sum_{j < 0} b_0 \cdot \text{im}(\Phi_{\mu_j}) \cdot t^j & \text{if } v_t(h) \geq 0. \end{cases}$$

Example 6.4. Let $F = \mathbb{C}$ and $t'/t = 1$, that is, t models $\exp(x)$. Let $h = 0$. Convention 4.4 and Remark 6.1 lead to the following table:

m	a_m	a_0	b_m	b_0	$\lambda_k, k \geq 0$	$\mu_l, l < 0$
0	0	0	1	1	k	l

From Example 4.2, we obtain $V_h^+ = \mathbb{C}$ and $V_h^- = \{0\}$. Thus, $\mathbb{C}$ is the auxiliary subspace V_h associated to $(F[t, t^{-1}], \mathcal{P}_h)$.

We have the following analogues of Proposition 5.5 and Definition 5.6.

Proposition 6.5. Let $f \in F[t, t^{-1}]^\times$, $k = \text{hdeg}(f)$, and $l = \text{tdeg}(f)$. Then

- (i) $f^+ = \mathcal{P}_h(p) + r$ for some $p \in F[t]$ with $\text{hdeg}(p) \leq k - m$ and $r \in V_h$,
- (ii) $f^- = \mathcal{P}_h(q) + s$ for some $q \in t^{-1} \cdot F[t^{-1}]$ with $\text{tdeg}(q) \geq l$ and $s \in V_h$.

Consequently, $F[t, t^{-1}] = \text{im}(\mathcal{P}_h) + V_h$.

Proof. (i) The proof is analogous to that of Proposition 5.5.

(ii) If $f^- = 0$, then $l \geq 0$. It suffices to set $q := 0$ and $s := 0$. Assume that $l < 0$ and the conclusion holds for all elements of $(t^{-1} \cdot F[t^{-1}])_{>l}$.

If $v_t(h) < 0$, then $f^- \equiv \mathcal{P}_h(a_0^{-1} f_l t^l) + g \pmod{F[t]_{<m}}$ for some $g \in (t^{-1} F[t^{-1}])_{>l}$ by the first congruence in (27). The conclusion holds by applying the inductive hypothesis to g .

Otherwise, $v_t(h) \geq 0$. Let (α, β) be an R-pair of $b_0^{-1} f_l$ with respect to Φ_{μ_l} . It follows from the second congruence in (27), $F[t]_{<m} \subset V_h$ and $(b_0 \beta) \cdot t^l \in V_h^-$ that $f^- \equiv \mathcal{P}_h(\alpha t^l) + g \pmod{V_h}$ for some $g \in (t^{-1} \cdot F[t^{-1}])_{>l}$. The conclusion holds by the inductive hypothesis.

By (i) and (ii), $f = \mathcal{P}_h(p + q) + r + s$ and $r + s \in V_h$. Hence, $F[t, t^{-1}] = \text{im}(\mathcal{P}_h) + V_h$. $\square$

Definition 6.6. Let f, p, q, r , and s be as in Proposition 6.5. We call $(p + q, r + s)$ an *auxiliary pair* of f with respect to $(F[t, t^{-1}], \mathcal{P}_h)$, or an *auxiliary pair* of f if $(F[t, t^{-1}], \mathcal{P}_h)$ is clear from context. In addition, we define $(0, 0)$ to be the auxiliary pair of 0.

Based on the congruences in (26), (27), and the proof of Proposition 6.5, an algorithm is developed for computing auxiliary pairs (see Algorithm A.7 in the appendix).

Corollary 6.7. Let $h \in F$ and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then, for every element $f \in F^\times$, an auxiliary pair of f with respect to $(F[t, t^{-1}], \mathcal{P}_h)$ is an R-pair of f with respect to Φ_h .

Proof. Note that λ_0 defined in Remark 6.1 (ii) is equal to h under the assumption $h \in F$. The remainder of the proof is analogous to that of Corollary 5.7. $\square$

6.2. Intersecting $\text{im}(\mathcal{P}_h)$ with the associated auxiliary subspace

In this subsection, V_h stands for the auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$, and J_h denotes $\text{im}(\mathcal{P}_h) \cap V_h$. We will derive a necessary and sufficient condition for $J_h = \{0\}$.

The next two lemmas connect J_h with the kernels of $\mathcal{H}_d$ and $\mathcal{T}_d$ in (24). We prove only the second assertion of each lemma; the first assertion follows analogously.

Lemma 6.8. Let $f \in F[t, t^{-1}]^\times$, $k = \text{hdeg}(f)$ and $l = \text{tdeg}(f)$. Assume that $\mathcal{P}_h(f) \in V_h$. Then

- (i) $\text{hc}(f) \in \ker(\mathcal{H}_k)$ if $k \geq 0$, and
- (ii) $\text{tc}(f) \in \ker(\mathcal{T}_l)$ if $l < 0$.

Proof. (ii) Assume that $l < 0$ and let $f_l = \text{tc}(f)$. Suppose that $\mathcal{T}_l(f_l) \neq 0$. By (25), $\mathcal{T}_l(f_l)t^l$ is the tail term of $\mathcal{P}_h(f)$. Since $\mathcal{P}_h(f) \in V_h$ and $l < 0$, Definition 6.3 implies that $\mathcal{T}_l(f_l)t^l \in V_h^-$. In particular, $V_h^- \neq \{0\}$. It follows from Definition 6.3 that $v_t(h) \geq 0$ and $\mathcal{T}_l(f_l) \in b_0 \cdot \text{im}(\Phi_{\mu_l})$. By Remark 6.1 (iv), $\mathcal{T}_l = b_0 \mathcal{R}_{\mu_l}$. Thus, $\mathcal{R}_{\mu_l}(f_l) \in \text{im}(\Phi_{\mu_l})$. Since

$$\text{im}(\mathcal{R}_{\mu_l}) \cap \text{im}(\Phi_{\mu_l}) = \{0\},$$

we derive that $\mathcal{T}_l(f_l) = 0$, which is a contradiction. $\square$

Lemma 6.9.

- (i) If $\ker(\mathcal{H}_k) \neq \{0\}$ for some $k \in \mathbb{N}_0$, then for all $\sigma \in \ker(\mathcal{H}_k)^\times$, there exists $p \in F[t]$ with $\text{hdeg}(p) = k$ and $\text{hc}(p) = \sigma$ such that $\mathcal{P}_h(p) \in V_h$. Moreover, for every $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists a constant $c \in C$ such that $f - cp \in t^{-1} \cdot F[t^{-1}]$.
- (ii) If $\ker(\mathcal{T}_l) \neq \{0\}$ for some $l \in \mathbb{N}^-$, then for all $\tau \in \ker(\mathcal{T}_l)^\times$, there exists $q \in t^{-1} \cdot F[t^{-1}]$ with $\text{tdeg}(q) = l$ and $\text{tc}(q) = \tau$ such that $\mathcal{P}_h(q) \in V_h$. Moreover, for every $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists a constant $c \in C$ such that $f - cq \in F[t]$.

Proof. (ii) Since $\ker(\mathcal{T}_l) \neq \{0\}$, Remark 6.1 (iii) implies that $v_t(h) \geq 0$. By (25) and $\tau \in \ker(\mathcal{T}_l)$, the image $\mathcal{P}_h(\tau t^l)$ belongs to $F[t]_{<m} + (t^{-1} \cdot F[t^{-1}])_{>l}$.

Let (g, r) be an auxiliary pair of $\mathcal{P}_h(\tau t^l)$. Then

$$\mathcal{P}_h(\tau t^l) = \mathcal{P}_h(g) + r, \quad g \in (t^{-1} \cdot F[t^{-1}])_{>l}, \quad \text{and} \quad r \in V_h$$

by Proposition 6.5 (ii). Set $q := \tau t^l - g$. Then $\text{tdeg}(q) = l$, $\text{tc}(q) = \tau$, and $\mathcal{P}_h(q) \in V_h$.

Assume that $f \in F[t, t^{-1}]$ with $d = \text{tdeg}(f)$ and $\mathcal{P}_h(f) \in V_h$. If $d \geq 0$, then we set $c := 0$. Otherwise, $\text{tc}(f) \in \ker(\mathcal{T}_d)$ by Lemma 6.8 (ii). It follows from Remark 6.2 that $d = l$. Since $\dim_C(\ker(\mathcal{T}_l)) = 1$, there exists a constant $c \in C$ such that $\text{tc}(f) = c\tau$. Then $\mathcal{P}_h(f - cq) \in V_h$ and $\text{tdeg}(f - cq) > l$. Hence, $\text{tdeg}(f - cq) \geq 0$ by Lemma 6.8 (ii) and Remark 6.2. $\square$

We now present a criterion for $J_h = \{0\}$.

Proposition 6.10. $J_h = \{0\}$ if and only if one of the following two conditions holds.

- (i) $h \in F$,
(ii) $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$.

In particular, $J_h = \{0\}$ if $v_\infty(h) < 0$ and $v_t(h) < 0$.

Proof. Assume that $J_h = \{0\}$ and $h \notin F$. Suppose that $\ker(\mathcal{H}_k)$ is nontrivial for some $k \in \mathbb{N}_0$. Then there exists a polynomial $p \in F[t]^\times$ such that $\mathcal{P}_h(p) \in V_h$ by Lemma 6.9 (i). Moreover, $\mathcal{P}_h(p) \neq 0$ by Lemma 4.3. Thus, J_h is nontrivial, a contradiction. The same argument, using Lemma 6.9 (ii) and Lemma 4.3, gives a contradiction if $\ker(\mathcal{T}_l)$ is nontrivial for some $l \in \mathbb{N}^-$.

To show the converse, we let $f \in F[t, t^{-1}]^\times$ with $\mathcal{P}_h(f) \in V_h$. It suffices to show $\mathcal{P}_h(f) = 0$.

First, we consider the case in which $h \in F$. Then $m = 0$, $b_0 = 1$, $v_\infty(h) \geq 0$ and $v_t(h) \geq 0$. It follows from Definition 6.3 that $V_h = \sum_{i \in \mathbb{N}_0} \text{im}(\Phi_{\lambda_i}) \cdot t^i + \sum_{j \in \mathbb{N}^-} \text{im}(\Phi_{\mu_j}) \cdot t^j$, where $\lambda_i = it'/t + a_0$ and $\mu_j = jt'/t + a_0$ by Remark 6.1 (ii) and (iv).

Write $f = \sum_{i=1}^k f_i t^i$ with $f_i \in F$. Set $f_i = 0$ for $i \notin \{1, \dots, k\}$. Since $\mathcal{P}_h(f) = f' + a_0 f$,

$$\mathcal{P}_h(f) = \sum_{i=0}^k \mathcal{R}_{\lambda_i}(f_i) t^i + \sum_{j=-1}^{-1} \mathcal{R}_{\mu_j}(f_j) t^j$$

by Lemma 2.2 (ii). Hence, $\mathcal{R}_{\lambda_i}(f_i) \in \text{im}(\Phi_{\lambda_i})$ for all $i \in \mathbb{N}_0$ and $\mathcal{R}_{\mu_j}(f_j) \in \text{im}(\Phi_{\mu_j})$ for all $j \in \mathbb{N}^-$. Thus, $\mathcal{P}_h(f) = 0$ by $\text{im}(\Phi_{\lambda_i}) \cap \text{im}(\Phi_{\lambda_i}) = \{0\}$ and $\text{im}(\mathcal{R}_{\mu_j}) \cap \text{im}(\Phi_{\mu_j}) = \{0\}$.

Next, we assume that $h \notin F$, $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$. Suppose that $\mathcal{P}_h(f) \neq 0$. If $k = \text{hdeg}(f) \geq 0$, then $f_k \in \ker(\mathcal{H}_k)^\times$ by Lemma 6.8 (i), a contradiction. So $f^+ = 0$. Thus, we may further assume that $\text{tdeg}(f) = l < 0$. It follows from Lemma 6.8 (ii) that $f_l \in \ker(\mathcal{T}_l)^\times$. We have reached another contradiction.

In particular, if $v_\infty(h) < 0$ and $v_t(h) < 0$, then both $\mathcal{H}_k$ and $\mathcal{T}_l$ are injective by Remark 6.1 (i) and (iii), respectively. Therefore, $J_h = \{0\}$ by the second condition. $\square$

We introduce the notion of types to characterize J_h .

Definition 6.11. The type of J_h is defined to be

- (i) 0 if either $h \in F$, or $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$,
- (ii) (k, σ) if $h \notin F$, $\sigma \in \ker(\mathcal{H}_k)^\times$ for some $k \in \mathbb{N}_0$ and $\ker(\mathcal{T}_l) = \{0\}$ for all $l \in \mathbb{N}^-$,
- (iii) (l, τ) if $h \notin F$, $\ker(\mathcal{H}_k) = \{0\}$ for all $k \in \mathbb{N}_0$ and $\tau \in \ker(\mathcal{T}_l)^\times$ for some $l \in \mathbb{N}^-$,
- (iv) $((k, \sigma), (l, \tau))$ if $h \notin F$, $\sigma \in \ker(\mathcal{H}_k)^\times$ for some $k \in \mathbb{N}_0$ and $\tau \in \ker(\mathcal{T}_l)^\times$ for some $l \in \mathbb{N}^-$.

The type of J_h can be computed by an algorithm for solving the parametric logarithmic derivative problem in F . The type is zero if and only if $J_h = \{0\}$ by Proposition 6.10. If it is nonzero, then k, l are unique by Remark 6.2, and each of σ and τ , when present, is unique up to multiplication by an element of $C^\times$. The type is subsequently fixed once and for all.

Example 6.12. Let $F = C$, $t = \exp(x)$ and $h = -t^2/(t^2 + t + 1)$. Convention 4.4 and Remark 6.1 lead to the following table:

m	a_m	a_0	b_m	b_0	$\mathcal{H}_k(z), k \geq 0$	$\mathcal{T}_l(z), l < 0$
2	-1	0	1	1	$(k-1)z$	lz

Hence, the type of J_h can be taken as $(1, 1)$.

6.3. Echelon sequences and induced complements

In contrast to the primitive setting, we have $\dim_C(J_h) \leq 2$, as stated in the following proposition.

Proposition 6.13.

- (i) $\dim_C(J_h) = 1$ if the type of J_h is equal to (i, κ) for some $i \in \mathbb{Z}$.
- (ii) $\dim_C(J_h) = 2$ if the type of J_h is equal to $((k, \sigma), (l, \tau))$ for some $k \in \mathbb{N}_0$ and $l \in \mathbb{N}^-$.

Proof. Since the type of J_h is nonzero in both (i) and (ii), we see that $h \in F(t) \setminus F$ by Proposition 6.10. Consequently, $\mathcal{P}_h$ is injective by Lemma 4.3.

(i) Let us consider the case in which $i < 0$. By Lemma 6.9 (ii), there exists $q \in t^{-1} \cdot F[t^{-1}]$ with $\text{tdeg}(q) = i$ such that $\mathcal{P}_h(q) \in V_h$ and for every $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$, there exists $c \in C$ such that $g := f - cq \in F[t]$. Then $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f), \mathcal{P}_h(q) \in V_h$. Suppose that $g \neq 0$. Let $k = \text{hdeg}(g)$. Then $k \geq 0$ by $g \in F[t]$. It follows from Lemma 6.8 (i) that $\ker(\mathcal{H}_k) \neq \{0\}$, a contradiction to the type of J_h . Hence, $g = 0$. Consequently, $\mathcal{P}_h(f) = c\mathcal{P}_h(q)$. Note that $\mathcal{P}_h(q) \neq 0$ by the injectivity of $\mathcal{P}_h$. Thus, $\dim_C(J_h) = 1$. Likewise, one shows that $\dim_C(J_h) = 1$ if $i \geq 0$.

(ii) By Lemma 6.9, there exist two elements $p \in F[t]$ with $\text{hdeg}(p) = k$ and $q \in t^{-1} \cdot F[t^{-1}]$ with $\text{tdeg}(q) = l$ such that $\mathcal{P}_h(p), \mathcal{P}_h(q) \in V_h$. Let $f \in F[t, t^{-1}]$ with $\mathcal{P}_h(f) \in V_h$. Then

$$g := f - c_p p \in t^{-1} \cdot F[t^{-1}]$$

for some $c_p \in C$ by Lemma 6.9 (i). Moreover, $\mathcal{P}_h(g) \in V_h$ by $\mathcal{P}_h(f), \mathcal{P}_h(p) \in V_h$. Then there exists $c_q \in C$ such that $g - c_q q \in F[t]$ by Lemma 6.9 (ii). On the other hand, $g - c_q q \in t^{-1} \cdot F[t^{-1}]$ since $g, q \in t^{-1} \cdot F[t^{-1}]$. Thus, $g - c_q q = 0$ because $F[t] \cap (t^{-1} \cdot F[t^{-1}]) = \{0\}$. It follows that $f = c_p p + c_q q$. Consequently, $\mathcal{P}_h(f) = c_p \mathcal{P}_h(p) + c_q \mathcal{P}_h(q)$. Since $p \in F[t]$ and $q \in t^{-1} \cdot F[t^{-1}]$, the images $\mathcal{P}_h(p)$ and $\mathcal{P}_h(q)$ are C -linearly independent by the injectivity of $\mathcal{P}_h$. We conclude that $\dim_C(J_h) = 2$. $\square$

As in Section 5, we let Θ be an effective basis of F and set $\Omega := \{\theta t^i \mid \theta \in \Theta, i \in \mathbb{Z}\}$, which is an effective basis of $F[t, t^{-1}]$. We are going to construct an echelon sequence of J_h . By convention, we set the echelon sequence to $\emptyset$ if the type of J_h is equal to 0.

Proposition 6.14. Assume that $J_h \neq \{0\}$.

- (i) If the type of J_h is equal to (i, κ) for some $i \in \mathbb{Z}$, then there exist $p \in F[t, t^{-1}]$ and $\omega \in \Omega$ such that $(p, \mathcal{P}_h(p), \omega)$ is an echelon sequence of J_h . Moreover, $V_h \cap \ker(\omega^*)$ is a complement of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.
- (ii) If the type of J_h is equal to $((k, \sigma), (l, \tau))$ for some $k \in \mathbb{N}_0$ and $l \in \mathbb{N}^-$, then there exist $p_1, p_2 \in F[t, t^{-1}]$ and $\omega_1, \omega_2 \in \Omega$ such that $(p_1, \mathcal{P}_h(p_1), \omega_1), (p_2, \mathcal{P}_h(p_2), \omega_2)$ form an echelon sequence of J_h . Moreover, $V_h \cap \ker(\omega_1^*) \cap \ker(\omega_2^*)$ is a complement of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.

Proof. (i) By Proposition 6.13 (i), $\dim_C(J_h) = 1$. Then $J_h = \text{span}_C\{\mathcal{P}_h(p)\}$ for some $p \in F[t, t^{-1}]$. Let d and γ be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively. We choose an element $\theta \in \Theta$ such that $\theta^*(\gamma) \neq 0$. Set $\omega := \theta t^d$. Then $(p, \mathcal{P}_h(p), \omega)$ is an echelon sequence of J_h . By Lemma 2.9, $V_h \cap \ker(\omega^*)$ is a complement of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$.

(ii) By Proposition 6.13 (ii), $\dim_C(J_h) = 2$. Then $J_h = \text{span}_C\{\mathcal{P}_h(p), \mathcal{P}_h(q)\}$ for some $p, q \in F[t, t^{-1}]$. Let d and γ be the head degree and head coefficient of $\mathcal{P}_h(p)$, respectively, and let $\theta_p \in \Theta$ with $\theta_p^*(\gamma) \neq 0$. Set

$$\omega_p := \theta_p t^d, c_p := \omega_p^*(\mathcal{P}_h(p)), c_q := \omega_p^*(\mathcal{P}_h(q)), \quad \text{and} \quad r := q - c_p^{-1} c_q p.$$

Then $\mathcal{P}_h(r) \in \ker(\omega_p^*)$, while $\{\mathcal{P}_h(r), \mathcal{P}_h(p)\}$ remains a basis of J_h . Let e and ρ be the head degree and head coefficient of $\mathcal{P}_h(r)$, respectively, and choose $\theta_r \in \Theta$ with $\theta_r^*(\rho) \neq 0$. Set $\omega_r := \theta_r t^e$. Then $(r, \mathcal{P}_h(r), \omega_r), (p, \mathcal{P}_h(p), \omega_p)$ is an echelon sequence of J_h . By Lemma 2.9, the intersection $V_h \cap \ker(\omega_r^*) \cap \ker(\omega_p^*)$ is a complement of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$. The proof is completed by setting $p_1 := r, \omega_1 := \omega_r, p_2 := p, \text{ and } \omega_2 := \omega_p$. $\square$

The proofs of Propositions 6.13 and 6.14 lead to an algorithm for constructing an echelon sequence of J_h (see Algorithm A.8 in the appendix).

Definition 6.15. Let E be the echelon sequence in Proposition 6.14 (i) or (ii). We call the corresponding complement of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$ the *complement induced by E* .

We now obtain analogues of Theorem 5.20 and Corollary 5.21.

Theorem 6.16. Assume that Hypothesis 4.1 holds and t is hyperexponential. Then there exists an algorithm that, for every t -normalized element $h \in F(t)$, constructs a complete reduction $\tilde{\Phi}_h$ for $(F(t), \mathcal{R}_h)$.

Proof. It suffices to construct a complement W of $\text{im}(\mathcal{P}_h)$ in $F[t, t^{-1}]$ by Proposition 3.6.

Recall that V_h denotes the auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$, and that J_h denotes $\text{im}(\mathcal{P}_h) \cap V_h$. If the type of J_h is equal to zero, then $J_h = \{0\}$. Thus, Proposition 6.5 gives

$$F[t, t^{-1}] = \text{im}(\mathcal{P}_h) \oplus V_h,$$

and we set $W := V_h$. Assume that the type of J_h is equal to (i, κ) for some $i \in \mathbb{Z}$. By Proposition 6.14 (i), J_h has an echelon sequence $(p, \mathcal{P}_h(p), \omega)$, and we take $W := V_h \cap \ker(\omega^*)$. Otherwise, Proposition 6.14 (ii) gives an echelon sequence $(p_1, \mathcal{P}_h(p_1), \omega_1), (p_2, \mathcal{P}_h(p_2), \omega_2)$ of J_h , and we set $W := V_h \cap \ker(\omega_1^*) \cap \ker(\omega_2^*)$.

An algorithm for computing R-pairs with respect to $\tilde{\Phi}_h$ can be obtained by the same procedure as in the last two paragraphs of the proof of Theorem 5.20. $\square$

Corollary 6.17. Let $h \in F$, $\tilde{\Phi}_h$ be the complete reduction for $(F(t), \mathcal{R}_h)$ in Theorem 6.16, and Φ_h be the complete reduction for $(F, \mathcal{R}_h)$ in Hypothesis 4.1. Then

- (i) $\tilde{\Phi}_h(f) = \Phi_h(f)$ for all $f \in F$, and

$$(ii) \operatorname{im}(\tilde{\Phi}_0) \subset \operatorname{im}(\Phi_0) \oplus (t^{-1} \cdot F[t^{-1}]) \oplus (t \cdot F[t]) \oplus S_t.$$

Proof. (i) By Proposition 6.10 and $h \in F$, we have $J_h = \{0\}$. The assertion is immediate for $f = 0$. For $f \neq 0$, it follows from Corollary 6.7 by the same argument used for the case $I_h = \{0\}$ in the proof of Corollary 5.21 (i).

(ii) The auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_0)$ is

$$\sum_{k \in \mathbb{N}} \operatorname{im}(\Phi_{\lambda_k}) \cdot t^k + \operatorname{im}(\Phi_{\lambda_0}) + \sum_{l \in \mathbb{N}^-} \operatorname{im}(\Phi_{\mu_l}) \cdot t^l$$

by $b_0 = 1$ and Definition 6.3. Moreover, $\lambda_0 = 0$ since $h = 0$. The rest of the proof is analogous to that of Corollary 5.21 (ii). $\square$

7. Completing the induction

In this section, let $F_0 = C$ and let $F_n = C(t_1, \dots, t_n)$ be a transcendental Liouvillian extension as in Definition 1.9. Then $F_i = C(t_1, \dots, t_i)$ is also a transcendental Liouvillian extension of C for all $i \in [n]$. In view of Remark 1.6, we state the main result of this paper as follows.

Theorem 7.1. *There exists an algorithm that, for every element $h \in F_n$, constructs a complete reduction $\Phi_{n,h}$ for $(F_n, \mathcal{R}_h)$.*

Proof. We proceed by induction on n . If $n = 0$, then the conclusion holds by Example 3.2.

Let $n > 0$. Assume inductively that, for every $\alpha \in F_{n-1}$, there is an algorithm that constructs a complete reduction $\Phi_{n-1,\alpha}$ for $(F_{n-1}, \mathcal{R}_\alpha)$, and define the map in Hypothesis 4.1 by $\alpha \mapsto \Phi_{n-1,\alpha}$.

For $h \in F_n$, let (ξ, η) be the canonical normal form of h . The inductive hypothesis implies that one can construct a complete reduction for $(F_n, \mathcal{R}_\xi)$ by either Theorem 5.20 or Theorem 6.16, depending on whether t_n is primitive or hyperexponential over F_{n-1} . Therefore, a complete reduction $\Phi_{n,h}$ for $(F_n, \mathcal{R}_h)$ is obtained from Proposition 3.4 (ii). The map in Hypothesis 4.1 is then defined by $h \mapsto \Phi_{n,h}$ for all $h \in F_n$. $\square$

In practice, we need to maintain certain initial data to ensure that choices of elements from a given set are made consistently by Remark 2.5 in (Du et al., 2025). Let $\Theta_0 = \{1\}$. For each $i \in [n]$, we have an effective basis Θ_i of F_i given in (Du et al., 2025, Section 2.2). We set up a table $\mathbb{T}$ indexed by (i, α) , where $i \in [n]$ and $\alpha \in F_i$ is t_i -normalized. The corresponding entry is an initial sequence when t_i is primitive and an echelon sequence when t_i is hyperexponential. See Definition A.2 for the notion of initial sequences. The table $\mathbb{T}$ records the data needed to realize the map in Hypothesis 4.1 over F_i for subsequent inductive steps. It is global and is updated whenever a complete reduction for $(F_i, \mathcal{R}_\alpha)$ is constructed. These considerations lead to an algorithm for computing R-pairs with respect to $\Phi_{n,h}$ given in Theorem 7.1. Below is an outline of the algorithm, in which we initialize the table $\mathbb{T}$ to be empty.

Outline 7.2. Given $f, h \in F_n$, compute an R-pair of f w.r.t. $\Phi_{n,h}$.

1. (*Base case*) If $n = 0$, then an R-pair of f w.r.t. $\Phi_{n,h}$ is $(f/h, 0)$ if $h \neq 0$, and $(0, f)$ if $h = 0$. The algorithm terminates. (*See Examples 3.2 and 4.2*)
2. (*Preprocessing*)

(2.1) Find the canonical normal form (ξ, η) of h w.r.t. t_n by Algorithm GKS in (Chen et al., 2025a).

(2.2) Compute $(g, r, s) \in F_n \times F_{n-1}(t_n) \times S_{t_n, \xi}$ s.t. $\eta f = \mathcal{R}_\xi(g) + r/b + s$ by Algorithm GKSR in (Chen et al., 2025a), where $b = \operatorname{den}(\xi)$ and $S_{t_n, \xi}$ is defined by (4). (*See (5)*)

(2.3) If $r = 0$, then an R-pair of f w.r.t. $\Phi_{n,h}$ is $(\eta^{-1}g, \eta^{-1}s)$. The algorithm terminates.

(*See Propositions 3.6 and 3.4*)

3. (*Recursion*) Assume that t_n is primitive (resp. hyperexponential) over F_{n-1} .

(3.1) Compute an auxiliary pair (p, q) of r w.r.t. $(F_{n-1}\langle t_n \rangle, \mathcal{P}_\xi)$ by Algorithm A.1 (resp. Algorithm A.7).

(3.2) If $q = 0$, then $(\eta^{-1}(g + p), \eta^{-1}s)$ is a required R-pair. The algorithm terminates.

(*See Propositions 5.5, 6.5, and Remark 3.7*)

(3.3) (*Updating $\mathbb{T}$ *) If (n, ξ) is already an index of $\mathbb{T}$, retrieve the corresponding initial sequence L (resp. echelon sequence E). Otherwise, compute an initial sequence L of I_ξ by Algorithm A.4 (resp. an echelon sequence E of J_ξ by Algorithm A.8), and store L (resp. E) in the entry of $\mathbb{T}$ indexed by (n, ξ) .

(3.4) (*Trivial intersection*) If $L = \emptyset$ (resp. $E = \emptyset$), then $(\eta^{-1}(g + p), \eta^{-1}(q/b + s))$ is a required R-pair of f w.r.t. $\Phi_{n,h}$. The algorithm terminates.

(*See Propositions 5.9, 6.10, and Remark 3.7*)

(3.5) (*Nontrivial intersection*) Using Algorithm A.6 (resp. Algorithm A.9), compute an element $\tilde{q} \in F_{n-1}\langle t_n \rangle$ and an element $\tilde{r} \in W$ such that $q = \mathcal{P}_\xi(\tilde{q}) + \tilde{r}$, where W is the complement of $\text{im}(\mathcal{P}_\xi)$ in $F_{n-1}\langle t_n \rangle$ induced by L (resp. E) (see Definition A.3 (resp. Definition 6.15)). A required R-pair of f w.r.t. $\Phi_{n,h}$ is $(\eta^{-1}(g + p + \tilde{q}), \eta^{-1}(\tilde{r}/b + s))$.

(*See Theorems 5.20, 6.16, and Remark 3.7*)

Remark 7.3. The complete reduction outlined above depends on algorithms for solving the logarithmic derivative recognition problem and the parametric logarithmic derivative problem. In practice, such algorithms (Raab, 2012, Section 4.3.1) can be implemented provided that a $\mathbb{Q}$ -basis of C is available.

The following two examples illustrate the procedure in Outline 7.2 step by step. The first example provides the details of Example 1.11.

Example 7.4. Determine whether $f = \frac{x}{1 + \exp(x)} \cdot \exp\left(\frac{x}{1 + \exp(x)}\right)$ is a derivative in the transcendental elementary extension $(\mathbb{C}(x, t, y), d/dx)$, where $t = \exp(x)$ and $y = \exp\left(\frac{x}{1 + \exp(x)}\right)$.

Note that $f = gy$, where $g = \frac{x}{1 + \exp(x)}$. To reduce the number of recursive calls, we set $h := y'/y$, which is equal to g' . By the same argument as in Example 3.1, it suffices to compute an R-pair of g with respect to the complete reduction $\Phi_{2,h}$ for $(\mathbb{C}(x, t), \mathcal{R}_h)$.

Initially, we set $F_0 = \mathbb{C}$, $F_1 = F_0(x)$, and $F_2 = F_1(t)$, and initialize the table $\mathbb{T}$ to be empty.

Since h is a derivative in F_2 , it is t -normalized. So $(\xi, \eta) = (h, 1)$ in step 2.1. By Convention 4.4 and since $h = \xi$, we have

$$\begin{array}{ccccccc} a & b & m & a_m & b_m & a_0 & b_0 \\ \hline 1 + (1-x)t & (1+t)^2 & 2 & 0 & 1 & 1 & 1 \end{array}$$

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b) + 0$, where $r = x + xt$. Step 2.3 is skipped since $r \neq 0$.

In step 3.1, Algorithm A.7 computes an auxiliary pair $(p, q) = (0, x + xt)$ of r , where r belongs to the auxiliary subspace V_h associated to $(F_1[t, t^{-1}], \mathcal{P}_h)$.

Step 3.2 is skipped since q is nonzero. In step 3.3, $(2, h)$ is not yet an index of $\mathbb{T}$. Set

$$J_h := \text{im}(\mathcal{P}_h) \cap V_h.$$

An echelon sequence of J_h is constructed as follows.

First, we obtain that the type of J_h is $((0, 1), (-1, 1))$ by solving two parametric logarithmic derivative problems. Second, Algorithm A.7 computes the auxiliary pairs of $\mathcal{P}_h(1)$ and $\mathcal{P}_h(t^{-1})$, which are $(0, -xt + t + 1)$ and $(0, -t - 1 - x)$, respectively. Third, selecting xt as a pivot for $\mathcal{P}_h(1)$ and t as a pivot for $\mathcal{P}_h(t^{-1})$, we obtain the echelon sequence

$$E : (t^{-1}, -t - 1 - x, t), (1, -xt + t + 1, xt)$$

of J_h . Finally, we store E in the entry indexed by $(2, h)$. Step 3.4 is skipped since $E \neq \emptyset$.

In step 3.5, Algorithm A.9 projects q to $\text{im}(\mathcal{P}_h)$ and the complement of $\text{im}(\mathcal{P}_h)$ induced by E . The respective projections are $\mathcal{P}_h(-1 - t^{-1})$ and 0. Then an R-pair of g with respect to $\Phi_{2,h}$ is $(-1 - t^{-1}, 0)$, that is, $g = \mathcal{R}_h(-1 - t^{-1})$. It follows from $h = y'/y$ and $f = gy$ that $f = ((-1 - t^{-1})y)'$. In functional notation, we have

$$\int f = -(1 + \exp(-x)) \cdot \exp\left(\frac{x}{1 + \exp(x)}\right).$$

In the next example, F_n is neither a primitive tower nor a hyperexponential one.

Example 7.5. Let $t = \log(x)$. Determine whether

$$f = \frac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2 + 1)} \cdot \exp\left(\int \frac{2x^2 - 2t}{xt^2 + x}\right)$$

is a derivative in $\mathbb{Q}(x, t, f)$, where the derivation is d/dx .

We rewrite f to make the intermediate expressions more compact.

Let $y = \exp\left(\int \frac{2x^2 - 2t}{xt^2 + x}\right)$. Then $f = gy$, where

$$g = \frac{(2x^3 + 2x^2 - 1)t - t^3 - t^2 - 2x^5 + 1}{x^2(t^2 + 1)}.$$

Moreover, $\mathbb{Q}(x, t, f) = \mathbb{Q}(x, t, y)$. By the same reasoning as in Example 3.1, f belongs to $\mathbb{Q}(x, t, y)'$ if and only if there exists an element $w \in \mathbb{Q}(x, t)$ such that $\mathcal{R}_h(w) = g$, where

$$h := \frac{y'}{y} = \frac{2x^2 - 2t}{xt^2 + x}.$$

So it suffices to compute an R-pair of g with respect to the complete reduction $\Phi_{2,h}$ for $(\mathbb{Q}(x, t), \mathcal{R}_h)$.

Let $F_0 = \mathbb{Q}$, $F_1 = F_0(x)$, and $F_2 = F_1(t)$. A straightforward verification shows that h is t -normalized. So $(\xi, \eta) = (h, 1)$ in step 2.1. By Convention 4.4 and since $\xi = h$, we have

a	b	m	a_m	b_m	a_0	b_0
$-2x^{-1}t + 2x$	$t^2 + 1$	2	0	1	$2x$	1

In step 2.2, we have $g = \mathcal{R}_h(0) + (r/b)$, where $r = -x^{-2}t^3 - x^{-2}t^2 + (2x + 2 - x^{-2})t - 2x^3 + x^{-2}$.

In step 3.1, we compute an auxiliary pair

$$(p, q) = (x^{-1}t, 2xt - 2x^3) \tag{28}$$

of r with respect to $(F_1[t], \mathcal{P}_h)$. During the auxiliary reduction, an R-pair of $\text{lc}_t(r)$ with respect to $\Phi_{1,0}$ is computed. This produces the initial sequence 1, (0, 1), (0, 0), 1, which is stored in the entry indexed by (1, 0).

Note that h is the same as the h in Example 5.19, where the last table gives an initial sequence L , whose entries are $L[1] = 1$, $L[2] = (0, x^{-1})$, $L[3] = (0, -2x^{-1})$, $L[4] = x^{-1}$, $L[5] = 2$,

$$L[6] = (t^2 - x^2 + 1, 2xt - 2x^3, xt), \quad L[7] = (1, -2x^{-1}t + 2x, x^{-1}t),$$

and $L[8] = (t, -x^{-1}t^2 + 2xt + x^{-1}, x^{-1}t^2)$. Step 3.3 is then completed by storing L in the entry indexed by $(2, h)$. Step 3.4 is skipped because $L \neq \emptyset$.

Let E be the echelon sequence determined by L . In step 3.5, we project q in (28) to $\text{im}(\mathcal{P}_h)$ and W , respectively, where W is the complement of $\text{im}(\mathcal{P}_h)$ induced by E . Since $\deg_t(q) = 1$, no further

terms of E need to be computed. Indeed, q is the second component of $L[6]$. Then the respective projections of q are $\mathcal{P}_h(t^2 - x^2 + 1)$ and 0. It follows that an R-pair of g with respect to $\Phi_{2,h}$ is $(x^{-1}t + t^2 - x^2 + 1, 0)$. In other words, $g = \mathcal{R}_h(x^{-1}t + t^2 - x^2 + 1)$, which implies that

$$f = ((x^{-1}t + t^2 - x^2 + 1)y)'$$

In functional notation, we have $\int f = \left(\frac{\log(x)}{x} + \log(x)^2 - x^2 + 1\right) \cdot \exp\left(\int \frac{2x^2 - 2\log(x)}{x\log(x)^2 + x}\right)$. The same integral can also be evaluated using the `int()` command with the option

`method=parallelrisch`

in MAPLE 2026 and the `Integrate[]` command in MATHEMATICA 14.3.

8. Applications

We present two applications of the complete reduction in Theorem 7.1 for the case $h = 0$, in which the Risch operator $\mathcal{R}_0$ is the derivation on a transcendental Liouvillian extension. The first application concerns elementary integration, and the second addresses reduction-based creative telescoping. Since we only consider derivations in this section, the following notational convention is adopted for brevity.

Convention 8.1. Let $F_0 = C$ and $F_n = C(t_1, \dots, t_n)$ be a transcendental Liouvillian extension. For all $i \in [n]_0$, Ψ_i stands for the complete reduction $\Phi_{i,0}$ for $(F_i, ')$ given in Theorem 7.1.

With the above convention, we see that Ψ_0 is the identity map of C and Ψ_n is a complete reduction for $(F_n, ')$. For elementary integration, we need to distinguish the primitive generators of F_n from the hyperexponential ones.

Set $\mathbb{P} := \{i \in [n] \mid t'_i \in F_{i-1}\}$ and $\mathbb{H} := [n] \setminus \mathbb{P}$. Moreover, define

$$P := \text{span}_C\{\Psi_{i-1}(t'_i) \mid i \in \mathbb{P}\} \quad \text{and} \quad H := \text{span}_C\{\Psi_{j-1}(t'_j/t_j) \mid j \in \mathbb{H}\}.$$

Remark 8.2. If $i \in \mathbb{P}$, the type of $\text{im}(\mathcal{P}_0) \cap U_0$ in $F_{i-1}[t_i]$ can be taken as 1 by Example 5.11.

Some useful properties of remainders are given in the following two technical lemmas.

Lemma 8.3. For $i \in [n]_0$ and $f \in F_i$, $\Psi_n(f) - \Psi_i(f) \in P$.

Proof. For all $k \in \{i, \dots, n-1\}$, Remark 8.2 and Corollary 5.21 imply that $\Psi_{k+1}(f) - \Psi_k(f) \in P$ if $k+1 \in \mathbb{P}$, while Corollary 6.17 implies that $\Psi_{k+1}(f) - \Psi_k(f) = 0$ if $k+1 \in \mathbb{H}$. Hence, $\Psi_n(f) - \Psi_i(f)$ belongs to P by $\Psi_n(f) - \Psi_i(f) = \sum_{k=i}^{n-1} (\Psi_{k+1}(f) - \Psi_k(f))$. $\square$

Recall that S_{t_i} stands for the set of t_i -simple elements in $F_{i-1}(t_i)$ for all $i \in [n]$. Set

$$S := S_{t_1} + \dots + S_{t_n},$$

which is a direct sum. We now extend Definition 3.4 in (Du et al., 2020).

Definition 8.4. An element of F_n is said to be *simple* if it belongs to S .

Lemma 8.5. For every element $s \in S$, we have $s - \Psi_n(s) \in P$.

Proof. Let $s = \sum_{i \in [n]} s_i$, where $s_i \in S_{t_i}$. By Proposition 3.6 and Remark 3.7, $\Psi_i(s_i) = s_i$ for all $i \in [n]$. Since $\Psi_n(s) = \sum_{i \in [n]} \Psi_n(s_i)$, we have $s - \Psi_n(s) = \sum_{i \in [n]} (\Psi_i(s_i) - \Psi_n(s_i))$. The lemma follows from Lemma 8.3. $\square$

Next, we define residues of elements in S without reference to any particular generator of F_n .

Definition 8.6. Let $s \in S$ be of the form $\sum_{i \in [n]} s_i$ with $s_i \in S_{t_i}$. A *residue* of s is a residue of some s_j with $j \in [n]$ at an irreducible normal polynomial of positive degree in $F_{j-1}[t_j]$.

Residues of simple elements are well-defined by $S = \oplus_{i \in [n]} S_{t_i}$ and Remark 2.4.

We are ready to generalize Theorem 5.3 in (Du et al., 2025) from primitive towers to transcendental Liouvillian extensions.

Theorem 8.7. With Convention 8.1, we further assume that C is algebraically closed. Then an element f of F_n has an elementary integral over F_n if and only if

- (i) there exists an element $s \in S$ such that $\Psi_n(f) - s \in H + P$, and
- (ii) all residues of s belong to C .

Proof. Assume that both (i) and (ii) hold. By (ii), applying Lemma 2.6 to each nonzero simple component of s shows that s has an elementary integral over F_n . In view of (i), it therefore suffices to show that every element of $H + P$ has an elementary integral over F_n . For $i \in \mathbb{H}$, we have $t'_i/t_i = u'_i + \Psi_{i-1}(t'_i/t_i)$ for some $u_i \in F_{i-1}$, which implies that $\Psi_{i-1}(t'_i/t_i)$ is the derivative of $\log(t_i) - u_i$. For $i \in \mathbb{P}$, we have $t'_i = v'_i + \Psi_{i-1}(t'_i)$ for some $v_i \in F_{i-1}$, yielding that $\Psi_{i-1}(t'_i) \in F'_n$. Accordingly, $\Psi_n(f)$ has an elementary integral over F_n , and so does f .

Conversely, assume that f has an elementary integral over F_n . Then there exists a C -linear combination g of logarithmic derivatives in F_n such that $f \equiv g \pmod{F'_n}$ by Theorem 5.5.2 in (Bronstein, 2005). Thus, $\Psi_n(f) = \Psi_n(g)$ by $\Psi_n(F'_n) = \{0\}$. It remains to show that (i) and (ii) hold for $\Psi_n(g)$. By repeatedly applying Lemma 2.5 and the logarithmic derivative identity, we find an element $s \in S$ such that all residues of s belong to C and $g - s$ is a C -linear combination of t'_i/t_i for $i \in \mathbb{H}$. Then $\Psi_n(g) - \Psi_n(s) \in H + P$ by Lemma 8.3. Thus, $\Psi_n(g) - s \in H + P$ by Lemma 8.5. Both (i) and (ii) hold. $\square$

To compute elementary integrals by the above theorem, we set $R_i := t_i \cdot F_{i-1}[t_i]$ if $i \in \mathbb{P}$, and $R_i := (t_i \cdot F_{i-1}[t_i]) + (t_i^{-1} \cdot F_{i-1}[t_i^{-1}])$ if $i \in \mathbb{H}$. Moreover, let $R = C + R_1 + \cdots + R_n$. This sum is direct, and so is $R + S$.

Proposition 8.8. With Convention 8.1 and the notation just introduced, we have $\text{im}(\Psi_n) \subset R \oplus S$.

Proof. We proceed by induction on n . For $n = 0$, $\text{im}(\Psi_0) = C = R$ by Example 3.2. The conclusion holds. Assume that $n > 0$ and that the conclusion holds for $n - 1$. Then

$$\text{im}(\Psi_n) \subset \text{im}(\Psi_{n-1}) + R_n + S_{t_n}$$

by Corollary 5.21 (ii) if $n \in \mathbb{P}$ and by Corollary 6.17 (ii) if $n \in \mathbb{H}$. It follows from the inductive hypothesis that $\text{im}(\Psi_n) \subset R + S = R \oplus S$. $\square$

Next, we outline an algorithm for computing elementary integrals over F_n . In the outline, assume that C is algebraically closed.

Outline 8.9. Given $f \in F_n$, determine whether f has an elementary integral over F_n , and compute such an integral if one exists.

1. Compute an R -pair $(g, \Psi_n(f))$ of f w.r.t. Ψ_n . If $\Psi_n(f) = 0$, then $\int f = g$ and the algorithm terminates.
2. By Proposition 8.8, we decompose

$$\bullet \Psi_n(f) = r_f + s_f,$$

- $\Psi_{i-1}(t'_i/t_i) = r_i + s_i$ if $i \in \mathbb{H}$, and $\Psi_{i-1}(t'_i) = r_i + s_i$ if $i \in \mathbb{P}$,

where $r_f, r_i \in R$ and $s_f, s_i \in S$.

- Let $z_1, \dots, z_n$ be constant indeterminates.

- Compute the augmented matrix $(M | v) \in \mathbb{C}^{k \times (n+1)}$ of a linear system in $z_1, \dots, z_n$ induced by the equation $r_f - \sum_{i \in [n]} z_i r_i = 0$.
- Using Algorithm 2.8 in (Du et al., 2025), compute the augmented matrix $(N | w) \in \mathbb{C}^{l \times (n+1)}$ of a linear system in $z_1, \dots, z_n$ induced by the condition that all residues of $s_f - \sum_{i \in [n]} z_i s_i$ are constants.
- Solve the combined linear system with augmented matrix

$$\left(\begin{array}{c|c} M & v \\ N & w \end{array} \right).$$

If the system has no solution, then f has no elementary integral over F_n by Theorem 8.7 and the fact that $R + S$ is direct. The algorithm terminates.

- Let $(c_1, \dots, c_n)$ be a solution of the combined linear system. Set $s := s_f - \sum_{i \in [n]} c_i s_i$. Then

$$\begin{aligned} \Psi_n(f) &= r_f + s_f \\ &= \left(r_f - \sum_{i=1}^n c_i r_i \right) + \left(s_f - \sum_{i=1}^n c_i s_i \right) + \sum_{j \in \mathbb{H}} c_j \Psi_{j-1} \left(\frac{t'_j}{t_j} \right) + \sum_{k \in \mathbb{P}} c_k \Psi_{k-1} (t'_k) \\ &= s + \sum_{j \in \mathbb{H}} c_j \Psi_{j-1} \left(\frac{t'_j}{t_j} \right) + \sum_{k \in \mathbb{P}} c_k \Psi_{k-1} (t'_k). \end{aligned}$$

Applying Lemma 2.6 to the nonzero simple components of s shows that s has an elementary integral over F_n . Let $(q_j, \Psi_{j-1}(t'_j/t_j))$ be an R-pair of t'_j/t_j for all $j \in \mathbb{H}$, and $(q_k, \Psi_{k-1}(t'_k))$ be an R-pair of t'_k for all $k \in \mathbb{P}$. It follows that

$$\int f = g + \int s + \sum_{j \in \mathbb{H}} c_j (\log(t_j) - q_j) + \sum_{k \in \mathbb{P}} c_k (t_k - q_k), \quad (29)$$

where s is integrated using its residues (Du et al., 2023).

Remark 8.10. Algorithm 2.8 in (Du et al., 2025), used in step 3.2, originates from Raab (2012). In practice, this algorithm can be implemented provided that a $\mathbb{Q}$ -basis of C is available.

Example 8.11. Let us integrate

$$f := \frac{(\log x + 1) \operatorname{Li}(x^c)}{x^{c+1}},$$

where c is a constant indeterminate and $\operatorname{Li}(x)$ is the logarithmic integral, i.e., $\operatorname{Li}(x)' = 1/\log x$.

Let C be the algebraic closure of $\mathbb{Q}(c)$, $t_1 = x$, $t_2 = x^c$, $t_3 = \log(x)$, and $t_4 = \operatorname{Li}(x^c)$. Then $t'_1 = 1$, $t'_2/t_2 = c/t_1$, $t'_3 = 1/t_1$, $t'_4 = t_2/(t_1 t_3)$, and $F_4 := C(t_1, t_2, t_3, t_4)$ is a transcendental Liouvillian extension. Moreover, $f = (t_3 t_4 + t_4)/(t_1 t_2)$.

- By the algorithm in Outline 7.2, an R-pair of f with respect to Ψ_4 is $(g, \Psi_4(f))$, where

$$g = \frac{ct_2 t_3 - ct_3 t_4 - (c+1)t_4}{c^2 t_2} \quad \text{and} \quad \Psi_4(f) = \frac{c+1}{c^2 t_1 t_3}.$$

2. Projecting remainders $\Psi_4(f)$, $\Psi_0(t'_1)$, $\Psi_1(t'_2/t_2)$, $\Psi_2(t'_3)$, and $\Psi_3(t'_4)$ with respect to $R \oplus S$, we obtain the following table:

	$\Psi_0(t'_1)$	$\Psi_1(t'_2/t_2)$	$\Psi_2(t'_3)$	$\Psi_3(t'_4)$	$\Psi_4(f)$
Proj. in R	$r_1 = 1$	$r_2 = 0$	$r_3 = 0$	$r_4 = 0$	$r_f = 0$
Proj. in S	$s_1 = 0$	$s_2 = c/t_1$	$s_3 = 1/t_1$	$s_4 = t_2/(t_1 t_3)$	$s_f = \Psi_4(f)$

3. Let z_1, z_2, z_3 , and z_4 be constant indeterminates. We make the following two ansatzes:

- (i) $r_f = z_1 r_1 + z_2 r_2 + z_3 r_3 + z_4 r_4$, and
(ii) all residues of $s_f - z_1 s_1 - z_2 s_2 - z_3 s_3 - z_4 s_4$ are constants.

These yield a linear system in z_1, z_2, z_3 , and z_4 with augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \end{array} \right).$$

The system has a solution $(0, 0, 0, 0)$. Hence, f has an elementary integral over F_4 .

4. Computing the residues of s_f yields that

$$\int f = \frac{1}{c} \log(x) + \frac{c+1}{c^2} \log(\log(x)) - \frac{c \log x + (c+1)}{c^2 x^c} \text{Li}(x^c).$$

Example 8.12. Let $F_0 = \mathbb{C}$ and $F_3 = F_0(t_1, t_2, t_3)$, where

$$\frac{t'_1}{t_1} = \sqrt{-1}, \quad t'_2 = \frac{\sqrt{-1}(t_1^2 + 1)}{t_1^2 - 1} \quad \text{and} \quad \frac{t'_3}{t_3} = \frac{t_2^2 - 1}{2\sqrt{-1}t_2}.$$

Then F_3 is a transcendental Liouvillian extension, where t_1, t_2 , and t_3 model

$$\exp(\sqrt{-1}x), \quad \log(\sin x), \quad \text{and} \quad \exp\left(\int \frac{\log(\sin x)^2 - 1}{2\sqrt{-1} \log(\sin x)}\right),$$

respectively. Note that no nonzero constant is a derivative in F_3 .

We apply the algorithm in Outline 8.9 to determine whether

$$f := \frac{(\sqrt{-1}t_2^2 - \sqrt{-1}t_1^2 t_2^2 - 2)t_3 + \sqrt{-1}t_1^2 t_2 + 3\sqrt{-1}t_2 - 2t_2}{2(t_1^2 - 1)t_2(t_3 + t_2)}$$

has an elementary integral over F_3 .

1. An R-pair of f with respect to Ψ_3 is $(0, f)$. Hence, $\Psi_3(f) = f$ and $f \notin F'_3$.
2. Projecting the corresponding remainders with respect to $R \oplus S$ yields that

	$\Psi_0(t'_1/t_1)$	$\Psi_1(t'_2)$	$\Psi_2(t'_3/t_3)$	$\Psi_3(f)$
Proj. in R	$r_1 = \sqrt{-1}$	$r_2 = \sqrt{-1}$	$r_3 = t_2/(2\sqrt{-1})$	r_f
Proj. in S	$s_1 = 0$	$s_2 = 2\sqrt{-1}/(t_1^2 - 1)$	$s_3 = -1/(2\sqrt{-1}t_2)$	s_f

with $r_f = \frac{t_2}{2\sqrt{-1}}$ and $s_f = \frac{1}{(1 - t_1^2)t_2} + \frac{\sqrt{-1}(t_1^2 t_2^2 + t_1^2 - t_2^2 + 3)}{2(t_1^2 - 1)(t_2 + t_3)}$.

3. Let z_1, z_2 , and z_3 be constant indeterminates. Making ansatzes analogous to those in step 3 of Example 8.11 leads to the linear system in z_1, z_2 , and z_3 whose augmented matrix is

$$\left(\begin{array}{ccc|c} \sqrt{-1} & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \sqrt{-1} & 1 \end{array} \right).$$

Since the system is inconsistent, we conclude that f has no elementary integral over F_3 .

We now turn to the second application of our complete reduction, namely creative telescoping. We consider the case in which the integrand depends on a shift variable k . Precisely, let $C = \mathbb{C}(k)$, where k is a constant indeterminate. Assume further that the shift operator $S_k : k \mapsto k + 1$ can be extended to F_n , and the extended operator commutes with the derivation on F_n . For every $f \in F_n$ and every $i \in \mathbb{N}_0$, one can compute an R-pair (g_i, r_i) of $S_k^i(f)$ with respect to Ψ_n . Then $r_0, r_1, \dots, r_i$ are C -linearly dependent if and only if there exists a recurrence operator $\mathcal{L} \in C[S_k]^\times$ of order at most i such that $\mathcal{L}(f) \in F_n$. This allows us to search for a telescoper for f up to a prescribed order. Unfortunately, we have not found any criterion for the existence of telescopers in such extensions. The construction of telescopers in primitive towers is discussed in (Du et al., 2025, Section 5.2).

We now provide the details for Example 1.12.

Example 8.13. For all $k \in \mathbb{N}$, compute

$$\int_0^{\frac{\pi}{2}} \cos(2kx) \log(\sin(x)) \, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin(2kx) \log(\sin(x)) \, dx.$$

Let $f(k, x) = \exp(2k\sqrt{-1}x) \log(\sin(x))$ and $A(k) = \int_0^{\frac{\pi}{2}} f(k, x) \, dx$. The integrand $f(k, x)$ is not a D-finite function over $\mathbb{C}(k, x)$. The two integrals are the real and imaginary parts of $A(k)$, denoted by $R(k)$ and $I(k)$, respectively.

Let $F_0 = \mathbb{C}(k)$ and $F_4 = F_0(t_1, t_2, t_3, t_4)$ with

$$t'_1 = 1, \quad \frac{t'_2}{t_2} = \sqrt{-1}, \quad t'_3 = \frac{\sqrt{-1}(t_2^2 + 1)}{t_2^2 - 1}, \quad \text{and} \quad \frac{t'_4}{t_4} = 2k\sqrt{-1}.$$

The generators t_1, t_2, t_3 , and t_4 model $x, \exp(\sqrt{-1}x), \log(\sin(x))$, and $\exp(2k\sqrt{-1}x)$, respectively. Then $f(k, x) = t_3 t_4$ and $f(k+1, x) = t_2^2 t_3 t_4$. Using Ψ_4 , we find R-pairs (g_0, r_0) and (g_1, r_1) of $f(k, x)$ and $f(k+1, x)$, respectively, where

$$(g_0, r_0) = \left(-\frac{\sqrt{-1}(2kt_3 - 1)t_4}{4k^2}, -\frac{t_4}{k(t_2^2 - 1)} \right)$$

and

$$(g_1, r_1) = \left(-\frac{\sqrt{-1}(2k^2 t_2^2 t_3 + 2kt_2^2 t_3 - kt_2^2 - 2k - 2)t_4}{4(k+1)^2 k}, -\frac{t_4}{(k+1)(t_2^2 - 1)} \right).$$

Since $(k+1)r_1 - kr_0 = 0$, we have

$$(k+1)f(k+1, x) - kf(k, x) = (k+1)g'_1 - kg'_0. \quad (30)$$

Although neither $\lim_{x \rightarrow 0^+} g_0$ nor $\lim_{x \rightarrow 0^+} g_1$ exists,

$$\lim_{x \rightarrow 0^+} ((k+1)g_1 - kg_0) = \frac{\sqrt{-1}(2k+1)}{4k(k+1)}.$$

Integrating both sides of (30) from 0 to $\pi/2$, we have

$$A(k+1) - \frac{k}{k+1}A(k) = \frac{\sqrt{-1}((-1)^k - 2k - 1)}{4k(k+1)^2}. \quad (31)$$

To compute $A(k)$ for all $k \in \mathbb{N}$, we first evaluate $A(1) = \int_0^{\frac{\pi}{2}} \exp(2\sqrt{-1}x) \log(\sin(x)) dx$. Its integrand is represented by $t_2^2 t_3$ in F_3 . Using Ψ_3 , we obtain the following R-pair of the integrand:

$$\left(-\frac{\sqrt{-1}}{2} t_2^2 t_3 + \frac{\sqrt{-1}}{4} t_2^2 + \frac{\sqrt{-1}}{2} t_3 - \frac{t_1}{2}, 0 \right).$$

Hence, the integrand belongs to F'_3 . It follows that $A(1) = -\pi/4 - \sqrt{-1}/2$.

Taking real and imaginary parts of (31) and using the value of $A(1)$, we obtain

$$\left\{ \begin{array}{l} R(k+1) - \frac{k}{k+1} R(k) = 0 \\ R(1) = -\frac{\pi}{4} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} I(k+1) - \frac{k}{k+1} I(k) = \frac{(-1)^k - 2k - 1}{4k(k+1)^2} \\ I(1) = -\frac{1}{2}. \end{array} \right.$$

The recurrence for $R(k)$ together with its initial condition implies that $R(k) = -\pi/(4k)$. Using the MATHEMATICA package `Sigma` (Schneider, 2007), we get

$$I(k) = \frac{1 - (-1)^k}{4k^2} - \frac{1}{2k} \left(\sum_{i=1}^k \frac{1 - (-1)^i}{i} \right) + \frac{c}{k}$$

for some $c \in \mathbb{C}$. With $I(1) = -1/2$, we find that

$$I(k) = \frac{1 - (-1)^k}{4k^2} - \frac{1}{2k} \left(\sum_{i=1}^k \frac{1 - (-1)^i}{i} \right).$$

Remark 8.14. The result $R(k) = -\pi/(4k)$ is given by Formula 4.384.7 in (Gradshteyn and Ryzhik, 2007). Integrals analogous to $I(k)$ are evaluated in (Lyons et al., 2002) and (Raab, 2012, Chapter 5).

9. Experiments

We present empirical results on in-field integration obtained using the complete reduction (CR) in Theorem 7.1 and the `int()` command in MAPLE without any options. Experiments were carried out with MAPLE 2026 on an iMac equipped with a 3.6 GHz Intel Core i9 processor and 16 GB of memory. Our MAPLE scripts for CR, the experimental data, and the timings are contained in a ZIP file named `ReductionPair.zip` at <https://risc.jku.at/sw/reductionpair/>.

The derivation on each differential field considered in this section is d/dx . Each integrand is the derivative of a rational fraction in a transcendental Liouvillian extension $\mathbb{Q}(t_1, \dots, t_n)$. The numerator and denominator of a dense (resp. sparse) fraction are generated using

```
randpoly([t1, t2, ..., tn], degree=d)
```

with the `dense` option (resp. with the default option, which produces sparse polynomials).

All integrands in our experiments are derivatives, because otherwise `int()` may attempt to find an antiderivative in a larger class of closed forms. For each degree d , five derivatives were integrated. The outputs of CR were rational functions with coprime numerators and denominators. Timings are reported in seconds of MAPLE CPU time. Computations were aborted after 3600 seconds. We marked an entry with “ $\int$ ” whenever `int()` returned an unevaluated integral. We verified the correctness of each output of CR.

In the first suite, we let $n = 3$, $t_1 = x$, $t_2 = \exp(x)$, and $t_3 = \log(\exp(x) + x)$. Fractions were dense in t_1, t_2 , and t_3 . The average timings are summarized in Table 2.

In the second suite, we let $n = 3$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, and $t_3 = \exp(x^2/2)$. Integrands were generated in the same way as in the first suite. The average timings are given in Table 3.

Table 2

Dense fractions in x , $\exp(x)$, and $\log(\exp(x) + x)$.

d	1	2	3	4	5	6	7	8	9	10
CR	0.01	0.03	0.02	0.16	0.70	2.20	8.01	22.73	61.27	>3600
int()	0.04	0.08	0.16	0.48	1.44	4.59	27.84	44.41	111.00	>3600

Table 3

Dense fractions in x , $\log(x^2 + 1)$, and $\exp(x^2/2)$.

d	1	2	3	4	5	6	7	8	9	10
CR	0.02	0.01	0.05	0.11	0.44	1.61	4.83	12.29	39.43	96.84
int()	0.04	0.09	0.23	0.74	2.04	6.83	19.78	66.33	176.75	>3600

In the third suite, we let $n = 4$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, $t_3 = \exp(x^2/2)$, and $t_4 = \exp(\exp(x^2/2))$. Rational fractions were chosen to be sparse because dense fractions became prohibitively large when $d > 3$. The average timings are given in Table 4.

Table 4

Sparse fractions in x , $\log(x^2 + 1)$, $\exp(x^2/2)$, and $\exp(\exp(x^2/2))$.

d	1	2	3	4	5	6	7	8	9	10
CR	0.01	0.02	0.04	0.07	0.15	0.97	2.42	4.40	30.53	104.30
int()	0.07	$\int$	0.50	$\int$	1.06	2.84	21.11	14.62	$\int$	219.41

We integrated the derivatives of dense polynomials in t_1, t_2, t_3 , and t_4 . The average timings are given in Table 5.

Table 5

Dense polynomials in x , $\log(x^2 + 1)$, $\exp(x^2/2)$, and $\exp(\exp(x^2/2))$.

d	11	12	13	14	15	16	17	18	19	20
CR	0.24	0.31	0.39	0.51	0.62	0.76	0.95	1.17	1.38	1.77
int()	8.82	13.32	19.64	30.13	53.06	83.45	141.59	257.31	386.16	1242.43

Finally, we let $n = 4$, $t_1 = x$, $t_2 = \log(x^2 + 1)$, $t_3 = \exp(x^2/2)$, and $t_4 = \exp(x \log(x^2 + 1))$. The average timings for integrating the derivatives of sparse fractions and dense polynomials are summarized in Tables 6 and 7, respectively.

Table 6

Sparse fractions in x , $\log(x^2 + 1)$, $\exp(x^2/2)$, and $\exp(x \log(x^2 + 1))$.

d	1	2	3	4	5	6	7	8	9	10
CR	0.01	0.02	0.05	0.28	0.41	0.19	20.45	0.14	3.43	518.15
int()	0.08	0.21	0.36	$\int$	$\int$	0.88	41.39	1.44	7.67	1449.17

Table 7

Dense polynomials in x , $\log(x^2 + 1)$, $\exp(x^2/2)$, and $\exp(x \log(x^2 + 1))$.

d	11	12	13	14	15	16	17	18	19	20
CR	0.40	0.54	0.67	0.81	1.04	1.27	1.54	1.88	2.22	2.68
int()	10.06	14.66	21.34	31.84	51.05	71.08	92.80	126.09	174.35	279.45

We did not compare CR with int() for computing elementary integrals, because such a comparison would also depend on the algorithm used to determine constant residues. Our MAPLE scripts

use an evaluation-based algorithm described in (Du et al., 2023), which outperforms resultant-based algorithms developed in the 1970s (Bronstein, 2005, Section 4.4).

10. Concluding remarks

In this paper, we have constructed a complete reduction for $(F, \mathcal{R}_h)$, where F is a transcendental Liouvillian extension, $h \in F$, and $\mathcal{R}_h$ is the Risch operator associated to h . The complete reduction directly furnishes an algorithm for in-field integration and leads to a new algorithm for computing elementary integrals over F .

At least three challenging problems remain for future research. The first is to extend the results of this paper from transcendental Liouvillian extensions to arbitrary Liouvillian extensions, in which some generators may be algebraic. The second is to develop a reduction-based algorithm for computing telescopers for elements in F without any a priori order bound. Solving this problem would require a criterion for the existence of telescopers for elements in F . The third concerns modifying the remainders produced by the complete reduction for algebraic functions in (Chen et al., 2016) to enable more efficient computation of elementary integrals over algebraic-function fields.

CRedit authorship contribution statement

Shaoshi Chen: Writing – original draft. **Hao Du:** Writing – original draft. **Yiman Gao:** Writing – original draft. **Hui Huang:** Writing – original draft. **Wenqiao Li:** Writing – original draft. **Ziming Li:** Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Shaoshi Chen and Ziming Li are Editorial Board Members for this journal and were not involved in the editorial review or the decision to publish this article.

Acknowledgements

We are grateful to the anonymous referees for their careful and constructive reviews. Their comments have helped us substantially improve the manuscript.

We thank Manuel Kauers and Clemens Raab for helpful discussions and valuable suggestions. Yiman Gao thanks the Computer Algebra and Applications group at RISC, led by Carsten Schneider, for many stimulating discussions.

Special thanks go to Junlin Xu of Maplesoft and Ralf Hemmecke of RISC for helping us carry out experiments with MAPLE 2026 and FRICAS, respectively.

Appendix A. Algorithm descriptions

This appendix is devoted to algorithmic descriptions of some constructive proofs in Sections 5 and 6. Let F be a differential field of characteristic zero, C be the constant subfield of F , and t be a regular monomial over F . For brevity, we use the phrase “the data in Convention 4.4” as shorthand for $F(t)$, h , m , a , b , a_m , b_m , a_0 , and b_0 throughout the algorithm specifications. For $\alpha \in F$, Φ_α stands for the complete reduction for $(F, \mathcal{R}_\alpha)$ in Hypothesis 4.1. In the appendix, Θ denotes an effective basis of F .

A.1. Algorithms in the primitive case

In this subsection, t is primitive over F , $h \in F(t)$ is t -normalized, U_h denotes the auxiliary subspace given in Definition 5.3, and $I_h := \text{im}(\mathcal{P}_h) \cap U_h$.

The first algorithm describes an auxiliary reduction in $F[t]$. It is based on (10) and the proof of Proposition 5.5.

Algorithm A.1. PRIMAUXREDINPUT: $f \in F[t]$ and the data in Convention 4.4OUTPUT: (g, r) , an auxiliary pair of f w.r.t. $(F[t], \mathcal{P}_h)$

1. $p \leftarrow f, (g, r) \leftarrow 0, 0, (d, p_d) \leftarrow \deg(p), \text{lc}(p)$
2. WHILE $d \geq m$ DO

IF $v_\infty(h) < 0$ THEN $(u, v) \leftarrow a_m^{-1} p_d t^{d-m}, 0$
 ELSE $(\alpha, \beta) \leftarrow$ an R-pair of p_d w.r.t. $\Phi_{a_m}, (u, v) \leftarrow \alpha t^{d-m}, \beta t^d$
 $(g, r) \leftarrow g + u, r + v, p \leftarrow p - \mathcal{P}_h(u) - v, (d, p_d) \leftarrow \deg(p), \text{lc}(p)$ (*by (10)*)
3. RETURN $(g, r + p)$

Next, we define a finite sequence L that uniquely determines an echelon sequence of I_h .

Definition A.2. Let $m = \text{Deg}(h)$ and λ be the type of I_h . We define an *initial sequence* L of I_h as follows.

1. If $\lambda = 0$, then set $L := \emptyset$.
2. Assume that $\lambda \neq 0$. Fix $(\tilde{\sigma}, \sigma)$ and $(\tilde{\tau}, \tau)$ as the first and second R-pairs associated to $(F[t], \mathcal{P}_h)$, respectively, and fix an element $\theta_\sigma \in \Theta$ such that $\theta_\sigma^*(\sigma) \neq 0$.

- 2.1. If $h \in F$, then set L to be the four-term sequence:

$$\lambda, (\tilde{\sigma}, \sigma), (\tilde{\tau}, \tau), \theta_\sigma. \quad (32)$$

- 2.2. Assume further that $h \in F(t) \setminus F$.

- 2.2.1. If $\theta_\sigma^*(i\sigma + \tau) \neq 0$ for all $i \in \mathbb{N}$, then set L to be the five-term sequence:

$$\lambda, (\tilde{\sigma}, \sigma), (\tilde{\tau}, \tau), \theta_\sigma, \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \quad (33)$$

where the last term is given by Corollary 5.18 (i).

- 2.2.2. If $\theta_\sigma^*(j\sigma + \tau) = 0$ for some $j \in \mathbb{N}$ but $j\sigma + \tau \neq 0$, then set L to be the seven-term sequence:

$$\lambda, (\tilde{\sigma}, \sigma), (\tilde{\tau}, \tau), \theta_\sigma, j, \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left(p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1}\right), \quad (34)$$

where the last two terms are given by Corollary 5.18 (ii).

- 2.2.3. If $j\sigma + \tau = 0$ for some $j \in \mathbb{N}$, then set L to be the $(j+6)$ -term sequence:

$$\lambda, (\tilde{\sigma}, \sigma), (\tilde{\tau}, \tau), \theta_\sigma, j, \left(q, \mathcal{P}_h(q), \theta t^d\right), \left(p_0, \mathcal{P}_h(p_0), \theta_0 t^{d_0}\right), \left\{\left(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{m+i-1}\right)\right\}_{i \in [j-1]}, \quad (35)$$

where the last $j+1$ terms are given by Corollary 5.18 (iii).

An initial sequence L with $L \neq \emptyset$ uniquely determines an echelon sequence E of I_h using (13), (14), and Algorithm A.1. For $i > 0$, p_i and $\mathcal{P}_h(p_i)$ are computed together using (13), (14), and Algorithm A.1, so no separate application of $\mathcal{P}_h$ to p_i is needed.

Definition A.3. Let L and E be given above with $L \neq \emptyset$. The complement of $\text{im}(\mathcal{P}_h)$ induced by E is also called the *complement induced by L* .

The next algorithm computes initial sequences.

Algorithm A.4. INITSEQ

INPUT: the data in Convention 4.4

OUTPUT: an initial sequence of I_h

1. $\lambda \leftarrow$ the type of I_h , IF $\lambda = 0$ THEN RETURN $\emptyset$ (* $I_h = \{0\}$ *)
2. $(\tilde{\sigma}, \sigma) \leftarrow$ a first R-pair associated to $(F[t], \mathcal{P}_h)$
 $(\tilde{\tau}, \tau) \leftarrow$ a second R-pair associated to $(F[t], \mathcal{P}_h)$
 $\theta_\sigma \leftarrow$ an element of Θ s.t. $\theta_\sigma^*(\sigma) \neq 0$, $L_0 \leftarrow \lambda$, $(\tilde{\sigma}, \sigma)$, $(\tilde{\tau}, \tau)$, θ_σ
3. IF $h \in F$ THEN RETURN L_0 (*by Corollary 5.17 and (32)*)
4. $p_0 \leftarrow \lambda$, $(d_0, l_0) \leftarrow \deg(\mathcal{P}_h(p_0))$, $\text{lc}(\mathcal{P}_h(p_0))$
 $\theta_0 \leftarrow$ an element of Θ s.t. $\theta_0^*(l_0) \neq 0$, $\omega_0 \leftarrow \theta_0 t^{d_0}$, $j \leftarrow -\theta_\sigma^*(\tau)/\theta_\sigma^*(\sigma)$, $k \leftarrow -\tau/\sigma$
5. IF $j \notin \mathbb{N}$ THEN RETURN $L_0, (p_0, \mathcal{P}_h(p_0), \omega_0)$ (*by Corollary 5.18 (i) and (33)*)
6. IF $k \notin \mathbb{N}$ THEN compute p_j and $\mathcal{P}_h(p_j)$ by (13), (14), and Algorithm A.1

$$l \leftarrow \text{lc}(\mathcal{P}_h(p_j)), \theta \leftarrow \text{an element of } \Theta \text{ s.t. } \theta^*(l) \neq 0$$

$$\text{RETURN } L_0, j, (p_0, \mathcal{P}_h(p_0), \omega_0), (p_j, \mathcal{P}_h(p_j), \theta t^{m+j-1})$$

(*by Corollary 5.18 (ii) and (34)*)

7. 7.1. compute $p_1, \mathcal{P}_h(p_1), \dots, p_{j-1}, \mathcal{P}_h(p_{j-1}), p_j, \mathcal{P}_h(p_j)$ by (13), (14), and Algorithm A.1
- 7.2. $(q, Q) \leftarrow p_j, \mathcal{P}_h(p_j)$
- 7.3. FOR i FROM $j-1$ TO 0 BY -1 DO

$$\text{IF } i > 0 \text{ THEN } \omega_i \leftarrow \theta_\sigma t^{m+i-1}$$

$$c \leftarrow \omega_i^*(\mathcal{P}_h(p_i))^{-1} \cdot \omega_i^*(Q), (q, Q) \leftarrow q - cp_i, Q - c\mathcal{P}_h(p_i)$$

$$7.4. (d, l) \leftarrow \deg(Q), \text{lc}(Q), \theta \leftarrow \text{an element of } \Theta \text{ s.t. } \theta^*(l) \neq 0$$

$$7.5. \text{RETURN } L_0, j, (q, Q, \theta t^d), (p_0, \mathcal{P}_h(p_0), \omega_0), (p_1, \mathcal{P}_h(p_1), \omega_1), \dots, (p_{j-1}, \mathcal{P}_h(p_{j-1}), \omega_{j-1})$$

(*by Corollary 5.18 (iii) and (35)*)

The correctness of this algorithm follows directly from Definition A.2 and Corollaries 5.17 and 5.18. Note that j and k in step 4 are equal whenever both are positive integers, which justifies the use of j in step 7.

Given an initial sequence L with $L \neq \emptyset$ and a positive integer l , we can compute the first l terms of the echelon sequence E by the following algorithm.

Algorithm A.5. PRIMECHSEQINPUT: $l \in \mathbb{N}$, the data in Convention 4.4, and an initial sequence L of I_h with $L \neq \emptyset$ OUTPUT: the first l terms of the echelon sequence induced by L

REMARK: In the following pseudo-code, $p_0 = L[1]$, and p_i and $\mathcal{P}_h(p_i)$ are computed by (13), (14), and Algorithm A.1 for $i > 0$.

1. $\lambda \leftarrow L[1]$, $(\tilde{\sigma}, \sigma) \leftarrow L[2]$, $(\tilde{\tau}, \tau) \leftarrow L[3]$, $\theta_\sigma \leftarrow L[4]$
2. IF $\text{len}(L) = 4$ THEN RETURN $\{(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{i-1})\}_{i \in [l]}$ (*by Corollary 5.17 and (32)*)
3. IF $\text{len}(L) = 5$ THEN RETURN $L[5]$, $\{(p_i, \mathcal{P}_h(p_i), \theta_\sigma t^{m+i-1})\}_{i \in [l-1]}$ (*by Corollary 5.18 (i) and (33)*)
4. $j \leftarrow L[5]$
5. IF $\text{len}(L) = 7$ AND $j\sigma + \tau \neq 0$ THEN

$$(q_1, Q_1, \omega_1) \leftarrow L[6]$$

$$\text{FOR } i \text{ FROM } 2 \text{ TO } l \text{ DO}$$

$$\text{IF } i \neq j+1 \text{ THEN } (q_i, Q_i, \omega_i) \leftarrow p_{i-1}, \mathcal{P}_h(p_{i-1}), \theta_\sigma t^{m+i-2} \text{ ELSE } (q_i, Q_i, \omega_i) \leftarrow L[7]$$

$$\text{RETURN } (q_1, Q_1, \omega_1), (q_2, Q_2, \omega_2), \dots, (q_l, Q_l, \omega_l) \quad (*by Corollary 5.18 (ii) and (34)*)$$

6. 6.1. FOR i FROM 1 TO $j + 1$ DO $(q_i, Q_i, \omega_i) \leftarrow L[5 + i]$
- 6.2. FOR i FROM $j + 2$ TO l DO $(q_i, Q_i, \omega_i) \leftarrow p_{i-1}, \mathcal{P}_h(p_{i-1}), \theta_\sigma t^{m+i-2}$
- 6.3. RETURN $(q_1, Q_1, \omega_1), (q_2, Q_2, \omega_2), \dots, (q_l, Q_l, \omega_l)$ (*by Corollary 5.18 (iii) and (35)*)

The following algorithm decomposes a given element of $F[t]$ into a component in $\text{im}(\mathcal{P}_h)$ and a component in its complement induced by an initial sequence.

Algorithm A.6. PRIMPROJ

INPUT: $f \in F[t]$, the data in Convention 4.4, and an initial sequence L of I_h with $L \neq \emptyset$

OUTPUT: $(g, r) \in F[t] \times W$ s.t. $f = \mathcal{P}_h(g) + r$, where W is the complement induced by L

1. $(g, r) \leftarrow$ an auxiliary pair of f computed by Algorithm A.1 (*auxiliary reduction*)
2. IF $r = 0$ THEN RETURN (g, r) (* the trivial projection*)
3. use Algorithm A.5 to compute the first k terms $(q_1, Q_1, \omega_1), \dots, (q_k, Q_k, \omega_k)$ of the echelon sequence E induced by L s.t. all remaining pivots of E have degree $> \deg(r)$
4. FOR i FROM k TO 1 BY -1 DO $c \leftarrow \omega_i^*(Q_i)^{-1} \omega_i^*(r)$, $(g, r) \leftarrow g + cq_i, r - cQ_i$ (*elimination*)
5. RETURN (g, r)

The first three steps of the above algorithm are evidently correct. Step 4 is justified by Lemma 2.13 and its proof. In step 5, r belongs to the complement of $\text{im}(\mathcal{P}_h)$ induced by L , since $\omega_i^*(r) = 0$ for all $i \in [k]$, while the pivot of the l th term does not occur in r for any $l > k$, by the choice of k in step 3.

A.2. Algorithms in the hyperexponential case

In this subsection, t is hyperexponential over F , $h \in F(t)$ is t -normalized, V_h is the auxiliary subspace given in Definition 6.3, and $J_h = \text{im}(\mathcal{P}_h) \cap V_h$. The elements λ_k and μ_l of F are defined in Remark 6.1 (ii) and (iv), respectively.

The first algorithm implements the auxiliary reduction in $F[t, t^{-1}]$. It is based on the congruences in (26) and (27) and the proof of Proposition 6.5.

Algorithm A.7. HYPEREXPAUXRED

INPUT: $f \in F[t, t^{-1}]$ and the data in Convention 4.4

OUTPUT: (g, r) , an auxiliary pair of f w.r.t. $(F[t, t^{-1}], \mathcal{P}_h)$

1. $p \leftarrow f^+$, $(k, p_k) \leftarrow \text{hdeg}(p)$, $\text{hc}(p)$, $q \leftarrow f^-$, $(l, q_l) \leftarrow \text{tdeg}(q)$, $\text{tc}(q)$, $(g, r) \leftarrow 0, 0$
2. WHILE $k \geq m$ DO

IF $v_\infty(h) < 0$ THEN $(u, v) \leftarrow a_m^{-1} p_k t^{k-m}, 0$, $g \leftarrow g + u$
 ELSE $(\alpha, \beta) \leftarrow$ an R-pair of p_k w.r.t. $\Phi_{\lambda_{k-m}}$, $(u, v) \leftarrow \alpha t^{k-m}, \beta t^k$, $(g, r) \leftarrow g + u, r + v$
 $p \leftarrow p - \mathcal{P}_h(u) - v$, $(k, p_k) \leftarrow \text{hdeg}(p)$, $\text{hc}(p)$ (*by (26)*)
3. WHILE $l < 0$ DO

IF $v_t(h) < 0$ THEN $(u, v) \leftarrow a_0^{-1} q_l t^l, 0$, $g \leftarrow g + u$
 ELSE $(\alpha, \beta) \leftarrow$ an R-pair of $b_0^{-1} q_l$ w.r.t. Φ_{μ_l} , $(u, v) \leftarrow \alpha t^l, b_0 \beta t^l$, $(g, r) \leftarrow g + u, r + v$
 $q \leftarrow q - \mathcal{P}_h(u) - v$, $(l, q_l) \leftarrow \text{tdeg}(q)$, $\text{tc}(q)$ (*by (27)*)
4. RETURN $(g, r + p + q)$

Next, we determine whether $J_h = \{0\}$ and find an echelon sequence of J_h with respect to $(F[t, t^{-1}], \mathcal{P}_h)$ if $J_h \neq \{0\}$. This algorithm is based on Propositions 6.13 and 6.14. In this algorithm, each nonzero type is represented as a sequence of one or two pairs.

Algorithm A.8. HYPEREXPECHSEQ

INPUT: the data in Convention 4.4

OUTPUT: $\emptyset$ if $J_h = \{0\}$; otherwise, an echelon sequence of J_h

1. $M \leftarrow$ the type of J_h
2. IF $M = 0$ THEN RETURN $\emptyset$ (* $J_h = \{0\}$ *)
3. (* $\dim_C(J_h) = 1$ *)
 IF $\text{len}(M) = 1$ THEN $(k, \sigma) \leftarrow M[1]$ (* retrieve the type *)
 $(g, r) \leftarrow$ the auxiliary pair of $\mathcal{P}_h(\sigma t^k)$ computed by Algorithm A.7
 $(p, P) \leftarrow \sigma t^k - g, r, \theta \leftarrow$ an element of Θ s.t. $\text{hc}(P) \notin \ker(\theta^*)$
 RETURN $(p, P, \theta t^{\text{hdeg}(P)})$ (* by Proposition 6.14 (i) *)
4. (* $\dim_C(J_h) = 2$ *)
 $(k, \sigma) \leftarrow M[1], (l, \tau) \leftarrow M[2]$ (* retrieve the type *)
 - 4.1. $(g_k, r_k) \leftarrow$ the auxiliary pair of $\mathcal{P}_h(\sigma t^k)$ computed by Algorithm A.7
 $(g_l, r_l) \leftarrow$ the auxiliary pair of $\mathcal{P}_h(\tau t^l)$ computed by Algorithm A.7
 $(p_k, P_k) \leftarrow \sigma t^k - g_k, r_k, (p_l, P_l) \leftarrow \tau t^l - g_l, r_l$ (* by Proposition 6.13 (ii) *)
 - 4.2. $d_k \leftarrow \text{hdeg}(P_k), \theta_k \leftarrow$ an element of Θ s.t. $\text{hc}(P_k) \notin \ker(\theta_k^*), \omega_k \leftarrow \theta_k t^{d_k}$
 $c \leftarrow \omega_k^*(P_k)^{-1} \omega_k^*(P_l), (q, Q) \leftarrow p_l - c p_k, P_l - c P_k, d \leftarrow \text{hdeg}(Q)$
 $\theta \leftarrow$ an element of Θ s.t. $\text{hc}(Q) \notin \ker(\theta^*), \omega \leftarrow \theta t^d$, RETURN $(q, Q, \omega), (p_k, P_k, \omega_k)$
(* by Proposition 6.14 (ii) *)

The last algorithm decomposes an element of $F[t, t^{-1}]$ into a component in $\text{im}(\mathcal{P}_h)$ and a component in the complement induced by an echelon sequence of J_h . It is simpler than Algorithm A.6, because an echelon sequence of J_h has at most two terms. We therefore present only its formal specification.

Algorithm A.9. HYPEREXP PROJINPUT: $f \in F[t, t^{-1}]$, the data in Convention 4.4, and an echelon sequence E of J_h with $E \neq \emptyset$ OUTPUT: $(g, r) \in F[t, t^{-1}] \times W$ s.t. $f = \mathcal{P}_h(g) + r$, where W is the complement induced by E **Data availability**

We have shared the link to our data/code at <https://risc.jku.at/sw/reductionpair/>.

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