Parallel Telescoping and Parameterized Picard–Vessiot Theory *

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ABSTRACT

Parallel telescoping is a natural generalization of differential creative-telescoping for single integrals to line integrals. It computes a linear ordinary differential operator L, called a parallel telescoper, for several multivariate functions, such that the application of L to the functions yields partial derivatives of a single function. We present a necessary and sufficient condition guaranteeing the existence of parallel telescopers for differentially finite functions, and develop an algorithm to compute minimal ones for compatible hyperexponential functions. Besides computing annihilators of parametric line integrals, we use the parallel telescoping for determining Galois groups of parameterized partial differential systems of first order.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algebraic Algorithms*

General Terms

Algorithms, Theory

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D-finite functions, Creative telescoping, Parallel telescopers, Parameterized Picard–Vessiot theory

1. INTRODUCTION

The problem of finding linear differential equations with polynomial coefficients for parametric integrals has a long history. It at least dates back to Picard [28] who proved the existence of such linear differential equations for integrals of algebraic functions involving parameters. His result has been generalized to higher-dimensional cases and led to Gauss–Manin connections [22, 23, 15]. The key for obtaining such linear differential equations is the method of *creative telescoping*, which was first formulated as an algorithmic tool by Zeilberger and his collaborators in the 1990s [35, 36, 34]. The method enables us to prove a large number of combinatorial identities in an automatic way [27]. For more recent developments, see the survey article [17].

Given a function $f(t, x)$ described by two homogeneous linear differential equations with polynomial coefficients in t and x, the method of differential creative-telescoping [1] finds a linear differential operator L in $\partial/\partial t$ with polynomial coefficients in t such that $L(f) = \partial g/\partial x$, where g is usually a linear combination of partial derivatives of f over the field of rational functions in t and x. The operator L is called a *telescoper* for f, and the function g is called a *certificate* of L. They can be used to evaluate parametric integrals of f with respect to x.

Recently, a connection has been revealed between differential creative-telescoping and Galois theory of parameterized differential equations in [9, 2, 10, 29, 13]. Consider a first-order partial differential system of the form:

$$\frac{\partial Y}{\partial x_1} = f_1, \ \ldots, \ \frac{\partial Y}{\partial x_n} = f_n, \tag{1}$$

where $f_1, \ldots, f_n$ are rational functions in t, $x_1, \ldots, x_n$ satisfying compatibility conditions. Its parameterized Galois group can be determined by constructing a linear ordinary differential operator L in $\partial/\partial t$ with polynomial coefficients in t such that

$$L(f_1) = \frac{\partial g}{\partial x_1}, \ldots, L(f_n) = \frac{\partial g}{\partial x_n}$$

for a single rational function g. The operator L will be referred to as a parallel telescoper for $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$. Parallel telescopers may also be used to evaluate parametric line integrals in the same manner as we do for single integrals by classical creative-telescoping.

In this paper, we present a necessary and sufficient condition guaranteeing the existence of parallel telescopers for differentially finite functions (see Definition 2). This condition can also be derived from the finiteness of the de Rham cohomology with coefficients in a holonomic D-module [3, Chapter 1, Theorem 6.1]. But our proof is elementary and constructive, which leads to a recursive

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method for computing parallel telescopers. The condition can easily be verified if the given functions are hyperexponential. We develop an algorithm to compute a parallel telescoper of minimal order for compatible hyperexponential functions. The algorithm can be used for constructing parallel telescopers for non-compatible ones, although its output may not be of minimal order. We also show how to determine the Galois group of a differential system of the form (1) by parallel telescoping.

The rest of the paper is organized as follows. In Section 2, we review the notion of differentially finite elements. In Section 3, we study the existence of parallel telescopers. We present an algorithm in Section 4 for constructing minimal parallel telescopers for hyperexponential functions. In Section 5, parallel telescoping is applied to determine Galois groups of parameterized partial differential systems of first order.

2. DIFFERENTIALLY FINITE ELEMENTS

Let k be an algebraically closed field of characteristic zero. Let δ_i be the usual partial derivative with respect to x_i on the field $k(x_1, \ldots, x_n)$ for all i with $1 \le i \le n$. For brevity, we set $\mathbf{x} = (x_1, \ldots, x_n)$. Over the differential field $(k(\mathbf{x}), \{\delta_1, \ldots, \delta_n\})$ there is a noncommutative algebra $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$ whose commutation rules are

$$D_i D_j = D_j D_i \quad \text{and} \quad D_i f = f D_i + \delta_i(f)$$

for all $i, j \in \{1, \ldots, n\}$ and $f \in k(\mathbf{x})$. The algebra is also called the ring of differential operators associated to $k(\mathbf{x})$. The commutation rules imply the following fact:

Fact 1. *Let $L \in k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$ and $[D_i, L] = D_i L - L D_i$ for every i with $1 \le i \le n$.*

(i) $[D_i, L] = 0$ if and only if L is free of x_i.

(ii) If L is free of D_i, then so is $[D_i, L]$.

(iii) If L is in $k[\mathbf{x}]\langle D_1, \ldots, D_n\rangle$, then the degree of $[D_i, L]$ in x_i is less than that of L.

Due to the noncommutativity of $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$, we make a convention that ideals, vector spaces, modules and submodules are all left ones in this paper.

Let M be a module over $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$. For an element $L \in k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$ and $h \in M$, the scalar product of L and h is denoted by $L(h)$. We say that L is an annihilator of h if $L(h) = 0$. The set of all annihilators of h is denoted by $\operatorname{ann}(h)$. This is an ideal in $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$.

Definition 2. *Let h be an element of a module over the ring $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$. We say that h is* differentially finite *(abbreviated as D-finite) over $k(\mathbf{x})$ if $\operatorname{ann}(h) \cap k(\mathbf{x})\langle D_i\rangle \neq \{0\}$ for all i with $1 \le i \le n$.*

It is straightforward to see that h is D-finite if and only if the submodule generated by h is a finite-dimensional linear space over $k(\mathbf{x})$. It follows that, if $h_1, \ldots, h_m$ are D-finite elements in a module over $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$, so is every element in the submodule generated by $h_1, \ldots, h_m$.

When a module consists of functions in $x_1, \ldots, x_n$, its D-finite elements are called D-finite functions. These are ubiquitous in combinatorics as generating functions. D-finite functions were first systematically investigated by Stanley in [31]. Their important algebraic properties have been revealed by Lipshitz in [20, 21]. We recall a lemma in [20, Lemma 3], which is the starting point of our study on parallel telescopers.

Lemma 3 (Lipshitz, 1988)**.** *If h is a D-finite element in a module over $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$, then*

$$\operatorname{ann}(h) \cap k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\langle D_i, D_j\rangle \neq \{0\}$$

for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

The next lemma allows one to remove redundant variables.

Lemma 4. *Let h be a D-finite element in a module over the ring $k(\mathbf{x})\langle D_1, \ldots, D_n\rangle$. If*

$$D_{m+1}(h) = D_{m+2}(h) = \cdots = D_n(h) = 0$$

for some $m \in \{1, \ldots, n-1\}$, then h is also a D-finite element over $k(x_1, \ldots, x_m)$.

Proof. By the definition of D-finite elements, it suffices to show that the intersection of $\operatorname{ann}(h)$ and $k(x_1, \ldots, x_m)\langle D_i\rangle$ is nontrivial for all i with $1 \le i \le m$. Suppose the contrary. Then, without loss of generality, we may further suppose that every nonzero annihilator of h in $k[\mathbf{x}]\langle D_1\rangle$ involves x_n. Among those annihilators, we choose one, say P, whose degree in x_n is minimal. By Fact 1 (i), $[D_n, P]$ is nonzero. By Fact 1 (ii), it belongs to $k(x_1, \ldots, x_n)\langle D_1\rangle$. Since both $D_n(h)$ and $P(h)$ are equal to zero, $[D_n, P]$ is also a nonzero annihilator of h in $k[\mathbf{x}]\langle D_1\rangle$. By Fact 1 (iii), it has degree in x_n less than that of P, a contradiction. ∎

3. PARALLEL TELESCOPERS

In this section, we define the notion of parallel telescopers for several multivariate functions in a module-theoretic setting, and study under what conditions parallel telescopers exist for D-finite elements.

3.1 Definition of parallel telescopers

In order to define parallel telescopers, we introduce a new indeterminate t, and extend the field $k(\mathbf{x})$ to $k(t, \mathbf{x})$, which is denoted by K, and set $\Delta = \{\delta_t, \delta_1, \ldots, \delta_n\}$, where δ_t stands for the usual partial derivative with respect to t on K. Moreover, let us denote by $\mathbf{R}$ the ring $K\langle D_t, D_1, \ldots, D_n\rangle$ of linear differential operators. The notions such as D-finite elements and annihilators carry over naturally to K and $\mathbf{R}$.

Definition 5. *Let $f_1, \ldots, f_n$ be in an $\mathbf{R}$-module. A nonzero operator L in $k(t)\langle D_t\rangle$ is called a* parallel telescoper *for f_1, ..., f_n with respect to $\mathbf{x}$ if there exists an element g in the submodule generated by f_1, ..., f_n over $\mathbf{R}$, such that*

$$L(t, D_t)(f_i) = D_i(g) \quad \text{for all } 1 \le i \le n.$$

The element g is called a certificate *of L with respect to $\mathbf{x}$.*

By Definition 5, the parallel telescopers for f_1, ..., f_n and zero form an ideal in the ring $k(t)\langle D_t\rangle$. The ideal is principal, since $k(t)\langle D_t\rangle$ is a left Euclidean domain. A generator of the ideal is called a *minimal parallel telescoper* for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$.

3.2 Existence of parallel telescopers

We derive a necessary and sufficient condition for the existence of parallel telescopers for D-finite elements. To this end, we need a differential analogue of [27, Thm. 6.2.1].

Lemma 6. *Let h be an element of an $\mathbf{R}$-module. If h is D-finite over K, then, for every $i \in \{1, \ldots, n\}$, there exists a nonzero operator $L_i \in k(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\langle D_t\rangle$ and an element g_i in the submodule generated by h such that $L_i(h) = D_i(g_i)$.*

Proof. Let N be the submodule generated by h over $\mathbf{R}$.

By Lemma 3, h has a nonzero annihilator in $K\langle D_t, D_i\rangle$, which is free of x_i. Among all of the x_i-free and nonzero annihilators for h, we choose one, say P_i, whose degree in D_i is minimal. If the degree d_i of P_i in D_i is equal to zero, then the lemma holds by taking $L_i = P_i$ and $g_i = 0$. Assume that $d_i > 0$. We can always write

$$P_i = L_i + D_i Q_i, \tag{2}$$

where L_i is in $k(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\langle D_t\rangle$, and Q_i is in $k(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\langle D_t, D_i\rangle$ whose degree in D_i is less than d_i.

Set $g_i := Q_i(h)$. Since $P_i(h) = 0$ and g_i is in N, it remains to show that L_i is nonzero in (2). Suppose that $L_i = 0$. Then $D_i(g_i) = 0$, which, together with the D-finiteness of g_i, implies that g_i is D-finite over $k(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ by Lemma 4. Thus, there exists a nonzero operator R_i in $k(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\langle D_t\rangle$ such that $R_i(g_i) = 0$. It follows that the product $R_i Q_i$ is also a nonzero and x_i-free annihilator of h. But it has degree in D_i less than d_i, which contradicts the minimality assumption for the degree of P_i in D_i. ■

To study the existence of parallel telescopers, we introduce the notion of compatible elements with respect to $\mathbf{x}$.

Definition 7. *The elements $f_1, \ldots, f_n$ of an $\mathbf{R}$-module are said to be* compatible *with respect to $\mathbf{x}$ if $D_i(f_j) = D_j(f_i)$ hold for all $1 \le i < j \le n$. These latter equations are called* the compatibility conditions *for $f_1, \ldots, f_n$.*

The following lemma shows that the compatibility conditions are sufficient for the existence of parallel telescopers for D-finite elements.

Lemma 8. *Let $f_1, \ldots, f_n$ be elements of an $\mathbf{R}$-module. If they are D-finite over K and compatible with respect to $\mathbf{x}$, then there exists a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$.*

Proof. Set $\mathbf{x}_m = (x_1, \ldots, x_m)$, and set $\mathbf{R}_m$ to be the subring $k(t, \mathbf{x}_m)\langle D_t, D_1, \ldots, D_m\rangle$ of $\mathbf{R}$ for all m with $1 \le m \le n$.

We proceed by induction on n. If $n=1$, then f_1 has a telescoper in $k(t)\langle D_t\rangle$ with respect to $\mathbf{x}_1$ by Lemma 6. Assume that the lemma holds for any $n-1$ elements that are both D-finite over $k(t, \mathbf{x}_{n-1})$ and compatible with respect to $\mathbf{x}_{n-1}$.

Assume that $f_1, f_2, \ldots, f_n$ are D-finite over $k(t, \mathbf{x}_n)$ and compatible with respect to $\mathbf{x}_n$. Denote by N the submodule generated by $f_1, \ldots, f_n$ over $\mathbf{R}_n$. By Lemma 6, there exists a nonzero operator L_n in $k(t, \mathbf{x}_{n-1})\langle D_t\rangle$ such that

$$L_n(f_n) = D_n(g_n) \quad \text{for some } g_n \in N. \tag{3}$$

Without loss of generality, we further assume that L_n in (3) is of minimal degree in D_t and is monic with respect to D_t.

First, we show that L_n belongs to $k(t)\langle D_t\rangle$. For all i with $1 \le i \le n-1$, we set $L_i = [D_i, L_n]$, which belongs to $k(t, \mathbf{x}_{n-1})\langle D_t\rangle$ by Fact 1 (ii), and has degree in D_t less than that of L_n, because L_n is monic with respect to D_t. Note that $L_n D_i(f_n) = L_n D_n(f_i) = D_n L_n(f_i)$, in which the first equality is immediate from the compatibility condition $D_i(f_n) = D_n(f_i)$, and the second from the fact that L_n is free of x_n. Thus, $L_i(f_n) = D_i L_n(f_n) - D_n L_n(f_i)$, which, together with (3), implies that

$$L_i(f_n) = D_i D_n(g_n) - D_n L_n(f_i) = D_n\left(\tilde{f}_i\right), \tag{4}$$

where $\tilde{f}_i := D_i(g_n) - L_n(f_i)$ for $i = 1, \ldots, n-1$. Since $\tilde{f}_i$ belongs to N, we see that $L_i = 0$, for otherwise, L_n would not be a nonzero operator that satisfies (3) and has minimal degree in D_t by (4). Thus, L_n is free of x_i by Fact 1 (i). Accordingly, $L_n \in k(t)\langle D_t\rangle$. Moreover, $L_i = 0$ and (4) imply

$$D_n(\tilde{f}_i) = 0 \quad \text{for all } i \text{ with } 1 \le i \le n-1. \tag{5}$$

Next, we apply the induction hypothesis to $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$. Since $f_1, \ldots, f_n$ are D-finite over $k(t, \mathbf{x}_n)$, so is g_n, and so is $\tilde{f}_i$ for all i with $1 \le i \le n-1$. By (5) and Lemma 4, $\tilde{f}_i$ is D-finite over $k(t, \mathbf{x}_{n-1})$. Moreover, $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ are compatible with respect to $\mathbf{x}_{n-1}$ because $f_1, \ldots, f_{n-1}$ are compatible with respect to $\mathbf{x}_{n-1}$ and because L_n is free of $\mathbf{x}_{n-1}$. Therefore, there exist a nonzero operator $\tilde{L} \in k(t)\langle D_t\rangle$ and an element $\tilde{g}$ in the submodule generated by $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ over $\mathbf{R}_{n-1}$ such that

$$\tilde{L}\left(\tilde{f}_i\right) = D_i(\tilde{g}) \text{ for } i \in \{1, \ldots, n-1\} \quad \text{and} \quad D_n(\tilde{g}) = 0. \tag{6}$$

The first equality in (6) is due to the induction hypothesis, and the second due to (5). Moreover, $\tilde{g}$ belongs to N.

At last, we verify that $\tilde{L}L_n$ is a parallel telescoper for f_1, $\ldots$, f_n. Set $g = \tilde{L}(g_n) - \tilde{g}$. It belongs to N because both g_n and $\tilde{g}$ do. For $i \in \{1, \ldots, n-1\}$, $\tilde{L}L_n(f_i) = \tilde{L}\left(D_i(g_n) - \tilde{f}_i\right)$ by the definition of $\tilde{f}_i$ in (4). It follows from $\tilde{L}D_i = D_i\tilde{L}$ and the first equality of (6) that, for all $i \in \{1, \ldots, n-1\}$,

$$\tilde{L}L_n(f_i) = D_i\tilde{L}(g_n) - D_i(\tilde{g}) = D_i\left(\tilde{L}(g_n) - \tilde{g}\right) = D_i(g).$$

Applying $\tilde{L}L_n$ to f_n, we get

$$\tilde{L}L_n(f_n) = \tilde{L}D_n(g_n) = D_n\left(\tilde{L}(g_n)\right) = D_n(g),$$

in which the first equality follows from (3) and the last from the second one in (6). Therefore, $\tilde{L}L$ is indeed a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}_n$. ■

The next theorem gives a necessary and sufficient condition on the existence of parallel telescopers for D-finite elements.

Theorem 9. *Let $f_1, \ldots, f_n$ be D-finite elements of an $\mathbf{R}$-module. Then they have a parallel telescoper with respect to $\mathbf{x}$ if and only if there exists a nonzero operator $P \in k(t)\langle D_t\rangle$ such that*

$$P(D_i(f_j) - D_j(f_i)) = 0 \quad \textit{for all } 1 \le i < j \le n. \tag{7}$$

Proof. Assume that $f_1, \ldots, f_n$ have a parallel telescoper P with respect to $\mathbf{x}$. Then there exists an element g in the submodule generated by $f_1, \ldots, f_n$ such that $P(f_i) = D_i(g)$ for all i with $1 \le i \le n$. Since $D_iD_j(g) = D_jD_i(g)$, we have $P(D_i(f_j) - D_j(f_i)) = 0$.

Conversely, assume that there exists a nonzero operator $P \in k(t)\langle D_t\rangle$ such that

$$P(D_i(f_j) - D_j(f_i)) = 0 \quad \text{for all } 1 \le i < j \le n.$$

Then $P(f_1), \ldots, P(f_n)$ are compatible, because P is free of $\mathbf{x}$. So there is a parallel telescoper L for $P(f_1), \ldots, P(f_n)$ by Lemma 8. Therefore, LP is a parallel telescoper for f_1, $\ldots$, f_n with respect to $\mathbf{x}$. ∎

4. HYPEREXPONENTIAL CASE

Let E be a differential field extension of (K, Δ), where K and Δ are as in Section 3. The set of extended derivations on E is also denoted by Δ. The derivations in Δ are assumed to commute with each other. Furthermore, we assume that k is the subfield of constants in E.

For an element $h \in E$ and an operator $L \in \mathbf{R}$ of the form

$$L = \sum_{i,j_1,\ldots,j_n \geq 0} a_{i,j_1,\ldots,j_n} D_t^i D_1^{j_1} \cdots D_n^{j_n}$$

with $a_{i,j_1,\ldots,j_n} \in K$, we define the application of L to h as

$$L(h) = \sum_{i,j_1,\ldots,j_n \geq 0} a_{i,j_1,\ldots,j_n} \delta_t^i \circ \delta_1^{j_1} \circ \cdots \circ \delta_n^{j_n}(h).$$

Then E is an $\mathbf{R}$-module whose multiplication is the application of an operator in $\mathbf{R}$ to an element of E.

A nonzero element $h \in E$ is said to be *hyperexponential* over K if the logarithmic derivative $\delta(h)/h$ belongs to K for all $\delta \in \Delta$. Hyperexponential functions are D-finite elements. In fact, the submodule generated by several hyperexponential functions over $\mathbf{R}$ is the linear space spanned by them. Two hyperexponential functions are said to be *similar* if their ratio belongs to K.

4.1 Determining the existence

The next proposition allows one to determine the existence of parallel telescopers for hyperexponential functions.

Proposition 10. *Let $h \in E$ be a hyperexponential function over K. Then $\operatorname{ann}(h) \cap k(t)\langle D_t \rangle \neq \{0\}$ if and only if the logarithmic derivative of h with respect to t is of the form*

$$\frac{\delta_t(p)}{p} + r \quad \text{for some } p \in k(\mathbf{x})[t] \text{ and } r \in k(t). \qquad (8)$$

Proof. Assume that $\delta_t(h)/h$ is of the form (8). Since p is a polynomial in t over $k(\mathbf{x})$, there exists a nonzero operator L in $k(t)\langle D_t \rangle$ annihilating p. It is easy to verify that $(D_t - r)(h/p) = 0$. Therefore, h is annihilated by a nonzero operator in $k(t)\langle D_t \rangle$. Such an operator is the tensor product (cf. [32, Corollary 2.19, Definition 2.20]) of L and $D_t - r$.

Conversely, assume that there exists a nonzero element L in $\operatorname{ann}(h) \cap k(t)\langle D_t \rangle$. Then $\delta_t(h)/h$ is a rational solution of the Riccati equation associated to L, although it does not have to be in $k(t)$. By formula (4.3) in [32, page 107],

$$\frac{\delta_t(h)}{h} = \frac{\delta_t(P)}{P} + Q + \frac{R}{S},$$

where P, Q, R and S are polynomials in t over the algebraic closure of $k(\mathbf{x})$, the roots of S are singular points of L, and the roots of P are nonsingular ones (see also [6, Theorem 1]). Moreover, one can assume that $\deg_t(R) < \deg_t(S)$ and that S is monic. Since the singular points of L are in k, the coefficients of S are in k as well. Following the algorithm for computing rational solutions of Riccati equations described in [6, § 4.3] or [32, Exercise 4.10], we see that R belongs to $k[t]$. The same conclusion holds for Q by the algorithm in [6, § 4.2], as Q is constructed by analyzing the pole of the associated Riccati equation at infinity. Set $r = Q + R/S$, which belongs to $k(t)$, and set $s = \delta_t(h)/h - r$, which is in $k(t, \mathbf{x})$ and equal to $\delta_t(P)/P$. Thus, the linear differential equation $\delta_t(Y) = sY$ has a polynomial solution P. Since s belongs to $k(t, \mathbf{x})$, the equation must have a polynomial solution p in $k(\mathbf{x})[t]$, which implies that $\delta_t(p)/p = s$. Then, the logarithmic derivative $\delta_t(h)/h$ is of the form (8). ∎

One can decide if the logarithmic derivative $\delta_t(h)/h$ in Proposition 10 is of the form (8) by computing its squarefree partial fraction decomposition with respect to t. A more efficient way is to apply Algorithm WeakNormalizer in [7, § 6.1] to $\delta_t(h)/h$, which delivers a polynomial p in $k(\mathbf{x})[t]$ such that the difference of $\delta_t(h)/h$ and $\delta_t(p)/p$ belongs to $k(t)$ if and only if $\delta_t(h)/h$ is of the form (8).

Let $h_1, \ldots, h_n$ be hyperexponential functions. Then h_1, $\ldots$, h_n have a parallel telescoper with respect to $\mathbf{x}$ if and only if, for every pair i, j with $1 \leq i < j \leq n$, there exists a nonzero operator $P_{i,j} \in k(t)\langle D_t \rangle$ such that

$$P_{i,j} \left(D_i(h_j) - D_j(h_i) \right) = 0. \qquad (9)$$

This is because the least common left multiple of the $P_{i,j}$ can be taken as the operator P in (7) of Theorem 9. For each pair (h_i, h_j), there are three cases to be considered: (i) If $D_i(h_j) = D_j(h_i)$, then set $P_{i,j} = 1$. (ii) If h_i is similar to h_j, then the difference $D_i(h_j) - D_j(h_i)$ is hyperexponential. So we can find $P_{i,j}$ by Proposition 10. (iii) If h_i is not similar to h_j, then (9) implies that both $P_{i,j}(D_i(h_j))$ and $P_{i,j}(D_j(h_i))$ are equal to zero. Proposition 10 is also applicable to the last case.

Example 11. *Consider the hyperexponential functions*

$$h_1 = \frac{t(x_1 + t + t^2 u)}{u(t + x_1)\sqrt{t}}, \quad h_2 = \frac{((t+1)^2 + x_1 x_2 + t(x_1 - 1))u - t x_1}{u(t + x_2)\sqrt{t}},$$

where $u := t + x_1 + x_2$. A direct calculation yields

$$h := D_2(h_1) - D_1(h_2) = -1/\sqrt{t}$$

The logarithmic derivative of h in t belongs to $k(t)$. Then $P := 2tD_t + 1$ is the operator in $k\langle D_t \rangle$ such that $P(h) = 0$. So h_1 and h_2 have a parallel telescoper with respect to x_1 and x_2 by Proposition 10.

4.2 Computing minimal parallel telescopers

This subsection is devoted to computing minimal parallel telescopers. First, we present a recursive algorithm, named ParaTele, for hyperexponential functions that are both compatible and similar. Next, we show that the algorithm can be easily adapted to compute minimal parallel telescopers for merely compatible hyperexponential functions.

Algorithm ParaTele: Given compatible functions $r_1 h$, $\ldots$, $r_n h$, where h is hyperexponential over K and r_1, $\ldots$, r_n are rational functions in K, compute a minimal parallel telescoper $L(t, D_t)$ for $r_1 h, \ldots, r_n h$ with respect to $\mathbf{x}$ and a certificate g of L.

1. Compute a minimal telescoper L_n for $r_n h$ with certificate g_n by the algorithms in [1, 5].

2. If $n = 1$, then return (L_n, g_n); otherwise, set

$$\tilde{f}_i := D_i(g_n) - L_n(r_i h) \text{ for } i = 1, \ldots, n-1.$$

3. Run ParaTele for functions $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ to get $(\tilde{L}, \tilde{g})$, where $\tilde{L}$ is of minimal order and $\tilde{g}$ is in the submodule generated by $\tilde{f}_i$'s over $k(t, x_1, \ldots, x_{n-1})\langle D_1, \ldots, D_{n-1}\rangle$.

4. Return $L := \tilde{L}L_n$ and $g := \tilde{L}(g_n) - \tilde{g}$.

Note that some of the rational functions $r_1, \ldots, r_n$ in Algorithm ParaTele may be equal to zero. So the input consists of either zero or similar hyperexponential functions. This guarantees that the recursion in step 3 can be executed.

It follows from the proof of Lemma 8 that Algorithm ParaTele always computes a parallel telescoper. Its minimality will be proved in Proposition 13. To this end, we need a lemma that plays a similar role for hyperexponential functions as Lemma 4 for D-finite ones.

Recall that $\mathbf{x}_m$ stands for $(x_1, \ldots, x_m)$ and $\mathbf{R}_m$ for the subring $k(t, \mathbf{x}_m)\langle D_t, D_1, \ldots, D_m\rangle$, where $m = 1, \ldots, n$.

Lemma 12. *Let $h_1, \ldots, h_m$ be hyperexponential elements of E. Assume that, for all i with $1 \le i \le m$,*

$$D_{m+1}(h_i) = \cdots = D_n(h_i) = 0. \tag{10}$$

Let N and N_m be the submodule generated by $h_1, \ldots, h_m$ over $\mathbf{R}$ and $\mathbf{R}_m$, respectively. If there exists a nonzero operator $T \in k(t)\langle D_t\rangle$ and $a \in N$ such that

$$T(h_i) = D_i(a) \quad \text{for all } i \text{ with } 1 \le i \le m,$$

then there exists $b \in N_m$ such that

$$T(h_i) = D_i(b) \quad \text{for all } i \text{ with } 1 \le i \le m.$$

In other words, T is a parallel telescoper for $h_1, \ldots, h_m$ with respect to $\mathbf{x}_m$.

Proof. Without loss of generality, assume that $\{h_1, \ldots, h_\ell\}$ is a maximal linearly-independent subset of $\{h_1, \ldots, h_m\}$ over K. Then $a = \sum_{j=1}^{\ell} a_j h_j$ for some $a_j \in K$, because N is the linear space spanned by $h_1, \ldots, h_\ell$ over K. Hence,

$$T(h_i) = \sum_{j=1}^{\ell} D_i(a_j h_j) = \sum_{j=1}^{\ell} (\delta_i(a_j) + a_j r_{i,j})\, h_j, \tag{11}$$

where $r_{i,j}$ stands for the logarithmic derivative $\delta_j(h_i)/h_i$ and i ranges from 1 to m. Then there exist $s_i \in \{1, \ldots, \ell\}$ and $w_{i,s_i} \in k(t, \mathbf{x}_m)$ such that

$$T(h_i) = w_{i,s_i} h_{s_i} \quad \text{for all } i \in \{1, \ldots, m\}.$$

In fact, s_i can be any integer between 1 and ℓ and w_{i,s_i} must be zero if $T(h_i) = 0$; and s_i is unique if $T(h_i)$ is nonzero by Proposition 4.1 in [19]. Thus, (11) can be rewritten as

$$w_{i,s} h_s = \sum_{j=1}^{\ell} (\delta_i(a_j) + a_j r_{i,j})\, h_j.$$

By the linear independence of $h_1, \ldots, h_\ell$, $T(h_i) = D_i(a)$ is equivalent to

$$\begin{cases} \delta_i(a_{s_i}) + a_s r_{i,s_i} = w_{i,s_i}, \\ \delta_i(a_j) + a_j r_{i,j} = 0 \text{ for } j \in \{1, \ldots, m\} \text{ with } j \ne s_i. \end{cases} \tag{12}$$

Let $\xi_{m+1}, \ldots, \xi_n \in k$ be such that $b_i = a_i(\mathbf{x}_m, \xi_{m+1}, \ldots \xi_n)$ is well-defined for all i with $1 \le i \le \ell$. Then (12) still holds if we replace a_i by b_i for $i = 1, \ldots, m$. This is because the substitution of $\xi_{m+1}, \ldots, \xi_n$ for $x_{m+1}, \ldots, x_n$ commutes with δ_i for all i with $1 \le i \le m$; and because both w_{i,s_i} and the $r_{i,j}$'s are free of $x_{m+1}, \ldots, x_n$ by (10).

Set $b = \sum_{j=1}^{\ell} b_j h_j$, which is in N_m. It follows from (12) that $T(h_i) = D_i(b)$ for all i with $1 \le i \le m$. ∎

We now prove the correctness of Algorithm ParaTele.

Proposition 13. *Let $h \in E$ be hyperexponential over K and $r_1, \ldots, r_n \in K$. If $r_1 h, \ldots, r_n h$ are compatible, then Algorithm ParaTele computes a minimal parallel telescoper for $r_1 h, \ldots, r_n h$ with respect to $\mathbf{x}$.*

Proof. The proof is based on induction on the number n of functions. Set $f_i = r_i h$ for $i = 1, \ldots, n$. Note that L_n and g_n obtained from step 1 can be identified with the telescoper and certificate in (3), respectively, because the $\mathbf{R}$-submodule generated by $f_1, \ldots, f_n$ is equal to that generated by h. Consequently, Algorithm ParaTele is just an algorithmic formulation of the proof of Lemma 8 with an additional assumption that $\tilde{L}$ is a minimal parallel telescoper for $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ with respect to $\mathbf{x}_{n-1}$. The conclusions made in the proof of Lemma 8 remain valid. In particular, $\tilde{L}L_n$ is a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$.

It remains to prove that $\tilde{L}L_n$ is of minimal order. Assume that $P \in k(t)\langle D_t\rangle$ is a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$. Then $P(f_i) = D_i(w)$ for all i with $1 \le i \le n$ and for some w in the submodule generated by $f_1, \ldots, f_n$ over $\mathbf{R}$. In particular, P is a telescoper for f_n with respect to x_n. Thus, $P = QL_n$ for some $Q \in k(t)\langle D_t\rangle$. Applying Q to $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ yields

$$Q(\tilde{f}_i) = QD_i(g_n) - P(f_i) = D_i(Q(g_n) - w) \tag{13}$$

for all i with $1 \le i \le n-1$. By (5) in the proof of Lemma 8, $D_n(\tilde{f}_i) = 0$ for all i with $1 \le i \le n-1$. So Lemma 12 implies that Q is a parallel telescoper for $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ with respect to $\mathbf{x}_{n-1}$. Thus, Q is a left multiple of $\tilde{L}$ obtained in step 3, because $\tilde{L}$ is a minimal parallel telescoper for $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$. So P is a left multiple of the product $\tilde{L}L_n$. Thus, $\tilde{L}L_n$ is of minimal order. ∎

Example 14. *Let h_1, h_2 and P be the same as in Example 11. Then $H_1 := P(h_1)$ and $H_2 := P(h_2)$ are compatible hyperexponential functions. Applying any telescoping algorithm in [1, 5] to H_1 yields a minimal telescoper*

$$L_1 := 4t^2 D_t^2 - 8tD_t + 5$$

for H_1 satisfying $L_1(H_1) = D_1(G_1)$ for some hyperexponential function G_1 over $k(t, x_1, x_2)$. Set $\tilde{H}_2 = L_1(H_2) - D_2(G_1)$. Then $L = L_2 L_1 := 8t^3 D_t^3 - 12t^2 D_t^2 + 18tD_t - 15$ is a minimal parallel telescoper for H_1 and H_2 with respect to x_1 and x_2.

Example 15. *Let $h_1 = 4x_1 h/x_2$ and $h_2 = (x_1^2 - t^2)h/x_2^2$, where $h = \exp\left(-(x_1^2 - t^2)^2/x_2^2\right)$. An easy calculation shows that h_1 and h_2 are compatible and similar. Applying ParaTele to h_1 and h_2 yields $L(h_i) = D_i(g)$ for $i = 1, 2$, where $L = D_t$ and $g = -4th/x_2$. Using this telescoping relation to evaluate the parametric line integral*

$$F(t, x_1, x_2) = \int_{(0,1)}^{(x_1,x_2)} h_1(t, x_1, x_2)\, dx_1 + h_2(t, x_1, x_2)\, dx_2,$$

we find that $D_t(F) = g(t, x_1, x_2) - g(t, 0, 1)$. It follows that

$$F(t, x_1, x_2) = \int_{-\infty}^{t} g(t, x_1, x_2)\, dt - \int_{-\infty}^{t} g(t, 0, 1)\, dt + c,$$

where c is a constant with respect to D_t. One can easily show that c is actually equal to zero by the facts $D_i(F)=h_i$ and $h_i(-\infty, 0, 1) = 0$ for $i = 1, 2$.

One can find more details, a Maple implementation of our algorithm and examples for parallel telescopers and parametric line integrals at

http://mmrc.iss.ac.cn/~zmli/ParaTele.html.

Let us consider how to compute a minimal telescoper for compatible hyperexponential functions $f_1, \ldots, f_n$. For simplicity, we assume that $f_1, \ldots, f_m$ and $f_{m+1}, \ldots, f_n$ form two distinct equivalence classes modulo similarity. The same idea applies to the case, in which there are more than two equivalence classes. Since f_i and f_j are not similar for all i with $1 \le i \le m$ and j with $m+1 \le j \le n$, the compatibility condition $D_i(f_j) = D_j(f_i)$ implies that $D_i(f_j)=D_j(f_i)=0$. Let P be a minimal telescoper for $f_1, \ldots, f_m$ over $\mathbf{x}_m$, and Q a minimal one for $f_{m+1}, \ldots, f_n$ with respect to $x_{m+1}, \ldots, x_n$. Then $P(f_i) = D_i(g)$ for all i with $1 \le i \le m$ and for some g in the submodule generated by $f_1, \ldots, f_m$ over $\mathbf{R}_m$, and $Q(f_j) = D_j(h)$ for all j with $m+1 \le j \le n$ and for some h in the submodule generated by $f_{m+1}, \ldots, f_n$ over $k(t, x_{m+1}, \ldots, x_n)\langle D_t, D_{m+1}, \ldots, D_n\rangle$. In particular, we have $D_i(h) = D_j(g) = 0$ for all i with $1 \le i \le m$ and j with $m+1 \le j \le n$.

Set L to be the least common left multiple of P and Q. Then there exist $U, V \in k(t)\langle D_t\rangle$ such that $L = UP = VQ$. A straightforward calculation implies that L is a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$. A certificate of L is $U(g) + V(h)$. Let L' be a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$. By Lemma 12, L' is a parallel telescoper for both $f_1, \ldots, f_m$ with respect to $\mathbf{x}_m$ and $f_{m+1}, \ldots, f_n$ with respect to $x_{m+1}, \ldots, x_n$. So it is a common left multiple of P and Q. Consequently, it is a left multiple of L. We conclude that L is a minimal telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$.

To construct a parallel telescoper for hyperexponential functions $f_1, \ldots, f_n$ that are not necessarily compatible with respect to $\mathbf{x}$, we compute a nonzero operator $P \in k(t)\langle D_t\rangle$ such that (7) holds. Then $P(f_1), \ldots, P(f_n)$ are compatible with respect to $\mathbf{x}$. Let L be a parallel telescoper for $P(f_1), \ldots, P(f_n)$. Then LP is a parallel telescoper for $f_1, \ldots, f_n$. But LP is not necessarily of minimal order.

5. PARAMETERIZED PICARD–VESSIOT THEORY

A generalized differential Galois theory having differential algebraic groups (as in [16]) as Galois groups was initiated in [18]. The parameterized Picard–Vessiot theory considered in [9] is a special case of the above generalized differential Galois theory and studies symmetry groups of the solutions of linear differential equations whose coefficients contain parameters. In this section, we show the connection of parallel telescoping with this parameterized theory.

Let F, containing $k(t)$ as a subfield, be a differentially closed field of characteristic zero, i.e., any consistent differential system with coefficients in F has solutions in F. Let $F(\mathbf{x})$ be the field of rational functions in $\mathbf{x}$. As before, Δ stands for the set $\{\delta_t, \delta_1, \ldots, \delta_n\}$ of derivations. In the following, $\mathrm{gl}_n(*)$ (resp. $\mathrm{GL}_n(*)$) denotes the set of $n \times n$ matrices (resp. invertible matrices) with entries in $*$.

Let E be a differential field extension of $F(\mathbf{x})$. For a subset $\Lambda \subset \Delta$, an element $c \in E$ is called a *Λ-constant* if $\lambda(c) = 0$ for all $\lambda \in \Lambda$. The set of all Λ-constants forms a subfield of E, which is denoted by C_E^Λ. Consider the differential system

$$D_1(Y) = A_1 Y, \ \ldots, \ D_n(Y) = A_n Y, \qquad (14)$$

where $A_i \in \mathrm{gl}_n(F(\mathbf{x}))$, the set of $n \times n$ matrices with entries in $F(\mathbf{x})$, such that

$$D_i(A_j) - D_j(A_i) = A_i A_j - A_j A_i.$$

As in the classical Galois theory, we now define the "splitting field" for the system (14).

Definition 16. *A* parameterized Picard–Vessiot extension *of $F(\mathbf{x})$ (PPV-extension of $F(\mathbf{x})$) for the system (14) is a Δ-field extension E of $F(\mathbf{x})$ satisfying*

(a) There exists a matrix $Z \in \mathrm{GL}_n(E)$ such that $D_i(Z) = A_i Z$ for all $i = 1, \ldots, n$ and E is generated as a Δ-field over $F(\mathbf{x})$ by the entries of Z.

(b) $C_E^\Lambda = C_{F(\mathbf{x})}^\Lambda = F$ for $\Lambda = \{\delta_1, \ldots, \delta_n\}$.

The parameterized Picard–Vessiot group *(PPV-group) associated with the PPV-extension E of $F(\mathbf{x})$ is the group*

$$Gal_\Delta(E/F(\mathbf{x})) = \{\sigma \in \mathrm{Aut}_{F(\mathbf{x})}(E) \mid \sigma\delta = \delta\sigma \text{ for } \delta \in \Delta\}.$$

The existence of PPV-extensions for parameterized differential systems has been established in [9, Thm 9.5 (1)] under the assumption that F is differentially closed. Recently, this existence result has been improved so that one only needs F to be algebraically closed [33] and under weaker closure conditions in [12]. In the classical Galois theory, the Galois group of an algebraic equation is a subgroup of the permutation group. In the non-parameterized differential case, the Galois group of a linear differential system is a linear algebraic group, i.e., a group of $n \times n$ matrices whose entries are elements in the field of constants satisfying certain polynomial equations. The PPV-group associated with a PPV-extension of $F(\mathbf{x})$ is a *linear differential algebraic group*, i.e., a group of $n \times n$ matrices whose entries are elements in F satisfying certain differential equations [9, Theorem 9.5 (2)].

Example 17 (Example 3.1 in [9])**.** *Consider the equation*

$$D_x(Y) = \frac{t}{x} Y.$$

The PPV-extension for this equation is the $\{\delta_t, \delta_x\}$-field, generated by the element $z = x^t$, i.e.,

$$E \triangleq F(x)(z, \delta_x(z), \delta_t(z), \ldots) = F(x, x^t, \log(x)).$$

The corresponding PPV-group is as follows:

$$Gal_\Delta(E/F(x)) = \{a \in F \mid a \neq 0 \text{ and } \delta_t\left(\frac{\delta_t(a)}{a}\right) = 0\}.$$

As a corollary of the general Galois correspondence [9, Theorem 9.5], the following lemma will be used frequently in the rest of this section.

Lemma 18. *Let E be a PPV-extension of $F(\mathbf{x})$ for some parameterized differential system and let $Gal_\Delta(E/F(\mathbf{x}))$ be the associated PPV-group. Then the set*

$$\{f \in E \mid \sigma(f) = f \text{ for all } \sigma \in Gal_\Delta(E/F(\mathbf{x}))\}$$

coincides with the field $F(\mathbf{x})$.

5.1 Galois groups of first-order systems

Unlike the usual Picard–Vessiot theory where we have a complete algorithm to compute the Galois group of a given linear differential equation over the field of rational functions [14], we have only partial algorithmic results for the PPV-theory. Algorithms for first and second order parameterized equations over $F(\mathbf{x})$, where $n = 1$, appear in [2, 10]. An algorithm to determine if a parameterized equation of arbitrary order has a unipotent PPV-group (or even certain kinds of extensions of such a group) as well as an algorithm to compute the group appears in [25]. An algorithm to determine if a parameterized equation has a reductive PPV-group and compute it if it does appears in [24].

We now show how one determines the PPV-group of a first-order differential system of the form

$$D_1(Y) = f_1, \ \ldots, \ D_n(Y) = f_n, \tag{15}$$

where $f_1, \ldots, f_n \in F(\mathbf{x})$ are compatible rational functions with respect to $\mathbf{x}$. Let E be the PPV-extension of $F(\mathbf{x})$ and let $z \in E$ be a solution of the system (15). For every $\sigma \in \mathrm{Gal}_\Delta(E/F(\mathbf{x}))$, $\sigma(z)$ is still a solution of the system (15). Then $\sigma(z){=}z{+}c_\sigma$ for some $c_\sigma \in C_E^\Lambda = F$ with $\Lambda = \{\delta_1, \ldots, \delta_n\}$. By fixing a solution z, we get a representation of the PPV-group $\mathrm{Gal}_\Delta(E/F(\mathbf{x}))$ as a subgroup of the additive group $(F, +)$. The differential subgroups of $(F, +)$ have been classified by Cassidy [8, Lemma 11] and Sit [30, Theorem 1.3, p.647]. That is, any subgroup G of $(F, +)$ is of the form $\{a \in F \mid L(t, D_t)(a) = 0\}$, where L is a linear differential operator in $F\langle D_t\rangle$. We call L the *defining operator* for G.

Lemma 19. *If the coefficients $f_1, \ldots, f_n$ of the system (15) are in $k(t)(\mathbf{x})$, then the defining operator L for its corresponding PPV-group is in $k(t)\langle D_t\rangle$.*

Proof. As noted above the PPV-group G can be identified with the set of solutions of an equation of the form $L(y){=}0$ where $L \in F\langle D_t\rangle$. To see that this group is actually defined over $k(t)$, note that $k(t, \mathbf{x})$ is a purely transcendental extension of $k(t)$ and so $k(t)$ is algebraically closed in $k(t, \mathbf{x})$. Furthermore, Δ consists of independent derivations over $k(t, \mathbf{x})$. Remark 2.9.2 and Theorem 2.8 of [12] imply that a parameterized Picard-Vessiot extension exists for our equations and Lemma 8.2 of [12] implies that the parameterized Picard-Vessiot group is defined over $k(t)$. Finally [30, Theorem 1.3, p.647] implies that this group is defined as claimed above. ∎

Now we present the main result of this section that the problem of determining the PPV-group of the system (15) with coefficients in $k(t, \mathbf{x})$ is equivalent to that of computing a minimal parallel telescoper for its coefficients.

Theorem 20. *Let $f_1, \ldots, f_n$ be the coefficients of the system (15) such that they are in $k(t, \mathbf{x})$ and compatible with respect to $\mathbf{x}$. Then $L \in k(t)\langle D_t\rangle$ is the defining operator for the PPV-group of the system (15) if and only if L is a minimal parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$.*

Proof. Let $\tilde{L}$ be the defining operator for the PPV-group G of the system (15). By Lemma 19, $\tilde{L}$ is in $k(t)\langle D_t\rangle$. We claim that $\tilde{L}$ is a parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$. Let $z \in E$ be a solution of (15). Then, for any $\sigma \in G$, $\sigma(z) = z + c_\sigma$, where $c_\sigma \in F$ is such that $\tilde{L}(c_\sigma) = 0$. For any $\sigma \in G$, $\sigma(\tilde{L}(z)) = \tilde{L}(\sigma(z)) = \tilde{L}(z + c_\sigma) = \tilde{L}(z) + \tilde{L}(c_\sigma) = \tilde{L}(z)$. Then $\tilde{g} := \tilde{L}(z) \in F(\mathbf{x})$ by Lemma 18. Since $\tilde{L}$ commutes with D_i for all $i = 1, \ldots, n$,

$$\tilde{L}(D_i(z)) = \tilde{L}(f_i) = D_i(\tilde{g}). \tag{16}$$

We now show that we can choose $\tilde{g} \in k(t, \mathbf{x})$ (and so $\tilde{L}$ will be a parallel telescoper with $\tilde{g} \in k(t, \mathbf{x})$). Since the f_i are in $k(t, \mathbf{x})$, equations (16) imply that $D_i(\tilde{g})$ belongs to $k(t, \mathbf{x})$. Expanding $\tilde{g}$ in partial fractions with respect to x_n and using induction, one sees that there is an element $c \in F$ such that $\tilde{g} - c \in k(t, \mathbf{x})$. Now let $L \in k(t)\langle D_t\rangle$ be a minimal parallel telescoper for $f_1, \ldots, f_n$ with respect to $\mathbf{x}$. Then, we have that L divides $\tilde{L}$. To complete the proof, it remains to show that $\tilde{L}$ divides L. It suffices to prove that $L(c_\sigma) = 0$ for any $\sigma \in G$. Since $L(f_i) = D_i(g)$ for some $g \in k(t, \mathbf{x}) \subset F(\mathbf{x})$,

$$D_i(L(z) - g) = L(D_i(z)) - D_i(g) = L(f_i) - D_i(g) = 0.$$

Therefore $L(z) - g \in F$ and so $L(z) \in F(\mathbf{x})$. For any $\sigma \in G$, $L(z) = \sigma(L(z)) = L(\sigma(z)) = L(z + c_\sigma) = L(z) + L(c_\sigma)$. Then $L(c_\sigma) = 0$. ∎

Example 21. *Consider the differential system*

$$D_1(Y) = f_1, \quad D_2(Y) = f_2, \tag{17}$$

where $f_1, f_2 \in k(t, x_1, x_2)$ are as follows:

$$f_1 = \frac{t}{x_1 + x_2 + t}, \quad f_2 = \frac{tx_2 + t^2 + x_1 + x_2 + t}{(x_1 + x_2 + t)(x_2 + t)}.$$

It is easy to check that $D_2(f_1) = D_1(f_2)$. Applying ParaTele *to f_1 and f_2 yields $L(f_i) = D_i(g)$ for $i = 1, 2$, where $L{=}D_t^2$. Then the corresponding PPV-group of system (17) is as follows: $\mathrm{Gal}_\Delta(E/F(\mathbf{x})) = \left\{a \in F \mid \delta_t^2(a) = 0\right\}$.*

5.2 An inverse problem

As in the classical Galois theory, a natural question that arises is the inverse problem: *Which groups occur as Galois groups over a given field?* In [9, Example 7.1], the authors consider a Δ-field $F(x)$, where $\Delta = \{\delta_t, \delta_x\}$ and F is a $\{\delta_t\}$-differentially closed field. They show that the additive group $(F, +)$ cannot be the Galois group of a parameterized Picard–Vessiot extension of this field. In the rest of this section, we show a similar result for fields of rational functions in several variables (for other results concerning the inverse problem, see [11, 24, 25, 26, 29]). The key tool will be the fact that parallel telescopers always exist for compatible rational functions. Let us first recall a lemma, which is an easy corollary of Theorem 4 of Chapter VII.3 in [16].

Lemma 22. *If G is the PPV-group of a PPV-extension E of $F(\mathbf{x})$, then $E = F(\mathbf{x})\langle z\rangle_\Delta$, satisfying for any $\sigma \in G$*

$$\sigma(z) = z + c_\sigma \text{ with } c_\sigma \in F.$$

Theorem 23. *The additive group $G = (F, +)$ is not the PPV-group of a PPV-extension of $F(\mathbf{x})$.*

Proof. We argue by contradiction. Assume that G is the PPV-group of some PPV-extension E of $F(\mathbf{x})$. Then Lemma 22 implies that $D_i(z) = f_i$ with $f_i \in F(\mathbf{x})$ for all $i = 1, \ldots, n$. Since D_i and D_j commute in $F(\mathbf{x})$, $f_1, \ldots, f_n$ are compatible. By Theorem 9, there exists L in $F\langle D_t\rangle$ such that $L(f_i) = D_i(g)$ for some $g \in F(\mathbf{x})$. By the same argument as in the proof of Theorem 20, $L(c_\sigma) = 0$ for all $\sigma \in G$. This implies that $G \subset \{c \in F \mid L(c) = 0\} \subsetneq F$, which is a contradiction with $G = F$. ■

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