An Additive Decomposition in Logarithmic Towers and Beyond

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ABSTRACT

We consider the additive decomposition problem in primitive towers and present an algorithm to decompose a function in a certain kind of primitive tower which we call S-primitive, as a sum of a derivative in the tower and a remainder which is minimal in some sense. Special instances of S-primitive towers include differential fields generated by finitely many logarithmic functions and logarithmic integrals. A function in an S-primitive tower is integrable in the tower if and only if the remainder is equal to zero. The additive decomposition is achieved by viewing our towers not as a traditional chain of extension fields, but rather as a direct sum of certain subrings. Furthermore, we can determine whether or not a function in an S-primitive tower has an elementary integral without the need to deal with differential equations explicitly. We also show that any logarithmic tower can be embedded into a particular extension where we can further decompose the given function. The extension is constructed using only differential field operations without introducing any new constants.

KEYWORDS

Additive decomposition, Primitive tower, Logarithmic tower, Symbolic integration, Elementary integral

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1 INTRODUCTION

Given a differential ring $(\mathcal{R}, \,')$ and an element $f \in \mathcal{R}$, we ask if the indefinite integral of f belongs to $\mathcal{R}$ and compute one if it does. In order to do this, we start with a decision problem stated as:

$$\text{Given } f \in \mathcal{R}, \text{ decide if } f \in \mathcal{R}', \text{ where } \mathcal{R}' := \{g' \mid g \in \mathcal{R}\}. \quad (1)$$

One can see that a positive answer to (1) tells us that a $g \in \mathcal{R}$ exists where $f = g'$ and then we proceed to compute such a g. The decision together with the computation is known as the integrability problem. If (1) produces a negative answer, then we say that f is *not integrable* in $\mathcal{R}$.

In the latter case, we would still like to be able to say something about the given function. Is there any information to help us understand how far off we are from being successful? The answer lies in the additive decomposition problem:

$$\text{Compute } g, r \in \mathcal{R} \text{ such that } f = g' + r,$$

where

(i) r is minimal in some sense;
(ii) $f \in \mathcal{R}'$ if and only if $r = 0$.

We call such an r a *remainder* of f in $\mathcal{R}$ and write

$$f \equiv r \mod \mathcal{R}'.$$

So, it is clear that an algorithm for solving the problem of additive decomposition also provides a solution to the integrability problem. Remainders may help us find "closed form" expressions for integrals of elements in $\mathcal{R}$, in the sense that the integrals belong to some extensions of $\mathcal{R}$. They also play an important role in reduction-based methods for creative telescoping.

The first additive decomposition due to Ostrogradsky [13] and Hermite [12] is for the differential field $\mathcal{F} = (\mathbb{C}(x), d/dx)$. Given a rational function $f \in \mathcal{F}$, they proposed an algorithm to compute the remainder $r \in \mathcal{F}$ of f such that r is proper and has a squarefree denominator, and r is minimal in the sense that if $f \equiv \tilde{r} \mod \mathcal{F}'$ for some $\tilde{r} \in \mathcal{F}$, then the denominator of r divides that of $\tilde{r}$.

There has been a rapid development of additive decompositions in both symbolic integration and summation [1, 3, 4, 6–9, 11, 16]. Most of the articles were motivated by computing telescopers based on reduction [2]. In the cited literature, some classes of functions that were studied include hyperexponential [3], algebraic [9], Fuchsian D-finite [7], and D-finite [16]. Additive decomposition problems in these classes have been fully solved. We observe that the ring of D-finite functions is not closed under composition or taking reciprocals. For example, $\log x$ is D-finite, but $\log(\log(x))$ and $1/\log(x)$ are not. In this paper, we consider a class of functions that is closed under these two operations.

Singer et al. in 1985 and then Raab in 2012 gave some decision procedures for finding elementary integrals in some Liouvillian extensions [14, 15] and in extensions which contain some nonlinear generators [14]. They recursively solve Risch differential equations until one of them has no solution, or else the integral can be found. In the implementation of Raab's algorithm, the former case outputs an integrable part and collects all nonzero terms that prevent the differential equations from having a solution. Recently, Chen, Du and Li [6] were able to construct remainders in some primitive extensions (they termed them "straight towers" and "flat towers") without the need to deal with differential equations explicitly.

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In this article, we expand their work [6] by developing a new algorithm to construct remainders for functions in "S-primitive towers" (see Definition 4.3), which may not be straight or flat. Instances for S-primitive towers include differential field extensions generated by finitely many logarithmic functions and logarithmic integrals.

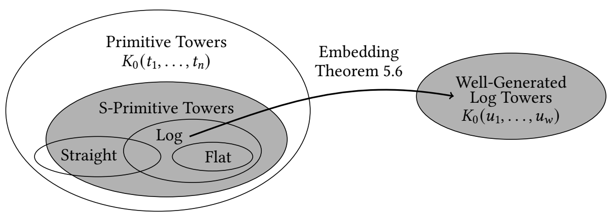

Figure 1: The gray ellipses on the left indicate the fields of functions for which we can construct a remainder. The embedding gives us a field extension ($n \le w$) where we can decompose functions further.

The organization of this article is as follows. In Sections 2 and 3, we give some relevant definitions associated to primitive towers, and then present a different way to view the towers. In Section 4, we give an algorithm for additive decompositions in S-primitive towers, and present a criterion for elementary integrability for the functions in such a field. In Section 5, we show how to construct a well-generated logarithmic tower to which a logarithmic tower can be embedded. Concluding remarks are given in Section 6.

2 PRELIMINARIES

Let K be a field of characteristic zero and $K(t)$ be the field of rational functions in t over K. An element of $K(t)$ is said to be *t-proper* if the degree of its denominator in t is higher than that of its numerator. In particular, zero is t-proper. For each $f \in K(t)$, there is a unique t-proper element $g \in K(t)$ and a unique polynomial $p \in K[t]$ with

$$f = g + p. \tag{2}$$

Let $'$ be a derivation on K. The pair $(K, ')$ is called a *differential field.* An element c of K is called a *constant* if $c' = 0$. The set of constants in K, denoted by C_K, is a subfield of K. Set

$$K' := \{f' \mid f \in K\},$$

which is a linear subspace over C_K. We call K' the *integrable subspace* of K.

Let (E, δ) be a differential field containing K. We say that E is a *differential field extension* of K if $\delta|_K = {}'$. The derivation δ is also denoted by $'$ when there is no confusion. For an element f of K, we call f a *logarithmic derivative* in K if $f = g'/g$ for some $g \in K \backslash \{0\}$. Let t be transcendental over K and $t' \in K[t]$, so that $p' \in K[t]$ for all $p \in K[t]$. A polynomial p in $K[t]$ is said to be *t-normal* if $\gcd(p, p') = 1$. By Theorem 3.2.2 in [5], $'$ can be uniquely extended to $K(t)$ such that $K(t)$ is a differential field extension of K. For $f \in K(t)$, we say that f is *t-simple* if it is t-proper and has a t-normal denominator.

We next define primitive and logarithmic generators, which are based on Definitions 5.1.1 and 5.1.2 in [5], respectively.

Definition 2.1. *Let $(K, ')$ be a differential field, and E be a differential field extension of K. An element t of E is said to be* primitive *over K if $t' \in K$. A primitive element t is called a* primitive generator *over K if it is transcendental over K and $C_{K(t)} = C_K$. Furthermore, a primitive generator t is called a* logarithmic generator *over K if t' is a C_K-linear combination of logarithmic derivatives in K.*

An immediate consequence of Theorem 5.1.1 in [5] is:

Proposition 2.2. *Let t be primitive over K. Then t is a primitive generator over K if and only if $t' \notin K'$. Assume that t is a primitive generator over K. Then $p \in K[t]$ is t-normal if and only if p is squarefree.*

For the rest of the section, assume that $(K, ')$ is a differential field, and that t is a primitive generator over K.

Remark 2.3. *Let p be a polynomial in $K[t]$. By Lemma 5.1.2 in [5], the degree of p' is equal to one less than the degree of p if the leading coefficient of p is a constant, otherwise their degrees are equal.*

By Theorem 5.3.1 in [5] and Lemma 2.1 in [6], for each $f \in K(t)$, there exists a unique t-simple element h such that

$$f \equiv h \mod \big(K(t)' + K[t]\big). \tag{3}$$

In the literature [6], h is referred to as the *Hermitian part* of f with respect to t. Thus, we will use the notation $\mathrm{hp}_t(f)$. It is easy to check that hp_t is a C_K-linear map on $K(t)$. Because of the uniqueness of Hermitian parts and Lemma 2.1 in [6], we have the following:

Lemma 2.4. *Let $f, g \in K(t)$. Then*

(i) $f \in K(t)' + K[t] \iff \mathrm{hp}_t(f) = 0$,
(ii) *f is t-simple* $\iff f = \mathrm{hp}_t(f)$, *and*
(iii) $f \equiv g \mod (K(t)' + K[t]) \iff \mathrm{hp}_t(f) = \mathrm{hp}_t(g)$.

The next two lemmas give some nice properties of proper elements and logarithmic derivatives.

Lemma 2.5. *If $f \in K(t)$ is t-proper, then $f - \mathrm{hp}_t(f) \in K(t)'$.*

Proof. Since t is a primitive generator over K, the derivative of a t-proper element of $K(t)$ is also t-proper. By (3), $f = \mathrm{hp}_t(f) + g' + p$ for some $g \in K(t)$ and $p \in K[t]$. Let r be the t-proper part of g. Thus, $f - \mathrm{hp}_t(f) - r' = p + (g - r)'$ whose left-hand side is t-proper and whose right-hand side is a polynomial in t. Thus, both sides must be zero. Consequently, $f - \mathrm{hp}_t(f) = r' \in K(t)'$. □

Lemma 2.6. *Let $f \in K(t)$ be a logarithmic derivative.*

(i) *f is t-proper $\iff$ f is t-simple.*
(ii) *There exists a t-simple logarithmic derivative $g \in K(t)$ and a logarithmic derivative $h \in K$ such that $f = g + h$.*

Proof. (i) The only thing we need to show is that the denominator of f is t-normal. By the logarithmic derivative identity [5, Theorem 3.1.1 (v)], the denominator of f is squarefree, which is also t-normal by Proposition 2.2.

(ii) By irreducible factorization and the logarithmic derivative identity, $f = \left(\sum_i m_i p_i'/p_i\right) + \alpha'/\alpha$, where $\alpha \in K$, $m_i \in \mathbb{Z}$, and $p_i \in K[t]$ are monic irreducible and pairwise coprime. Then each p_i'/p_i is t-simple by Remark 2.3 and (i). We get (ii) by setting $g = \sum_i m_i p_i'/p_i$ and $h = \alpha'/\alpha$. □

The following lemma will be useful when we construct our remainders. This is the same as Lemma 2.3 in [6] and can also be found in [5].

Lemma 2.7. *Let $p \in K[t]$. If $p \in K(t)'$, then the leading coefficient of p is equal to $ct' + b'$ for some $c \in C_K$ and $b \in K$. As a special case, if $p \in K \cap K(t)'$, then $p \equiv ct' \mod K'$.*

3 MATRYOSHKA DECOMPOSITIONS

We denote $\{1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, n\}$ by $[n]$ and $[n]_0$, resp. Let K_0 be a field. For each $i \in [n]$, let $K_i = K_{i-1}(t_i)$, where t_i is transcendental over K_{i-1}. Then we have a tower of field extensions:

$$\begin{array}{ccccccc} K_0 & \subset & K_1 & \subset & \cdots & \subset & K_n \\ & & \shortparallel & & & & \shortparallel \\ & & K_0(t_1) & \subset & \cdots & \subset & K_{n-1}(t_n). \end{array} \tag{4}$$

We use $K_0(\bar{t})$ to denote the tower (4) generated by $\bar{t} := (t_1, \ldots, t_n)$.

For each $i \in [n]$, an element of K_n from (4) is said to be t_i*-proper* if it is free of $t_{i+1}, \ldots, t_n$ and the degree of its numerator in t_i is lower than that of its denominator. Let T_i denote the multiplicative monoid generated by $t_{i+1}, \ldots, t_n$ for all i with $0 \le i < n$, and set $T_n = \{1\}$. For each $i \in [n]$, let P_i be a non-unital subring of $K_i[t_{i+1}, \ldots, t_n]$ consisting of all the linear combinations of the elements of T_i whose coefficients are t_i-proper. Furthermore, let $P_0 = K_0[t_1, \ldots, t_n]$. A routine induction based on (2) shows

$$K_n = \bigoplus_{i=0}^{n} P_i. \tag{5}$$

Accordingly, we can not only view the tower as a chain of field extensions as described in (4), but also as a direct sum of rings as given in (5). The former enables us to describe a function recursively, and the latter helps us to decompose it additively.

Let π_i be the projection from K_n onto P_i with respect to (5). For every $f \in K_n$, we say $\pi_i(f)$ is the i*-th projection* of f, and write

$$f = \sum_{i=0}^{n} \pi_i(f),$$

which we call the *matryoshka decomposition* of f. Figure 2 illustrates this namesake. The property $P_i \cap \bigoplus_{i \ne j} P_j = \{0\}$ indicates zero as the (only) point of intersection, and is represented by a single dot in Figure 2. Viewing our towers in this way not only affords us a nice pictorial representation, but also allows us to describe the following ordering (which will later be used to define a remainder).

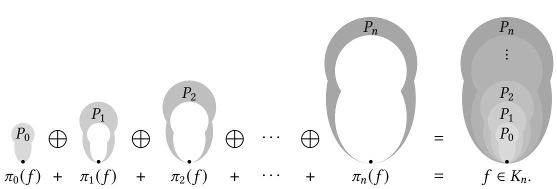

Figure 2: Matryoshka Decomposition

Example 3.1. *Let $f = (t_2 + x)(t_3^2 - t_1 t_3 + x t_2)/(x t_2 t_3)$ be in K_3 with $K_0 = \mathbb{Q}(x)$. Then the matryoshka decomposition of f is*

$$\pi_0(f) + \pi_1(f) + \pi_2(f) + \pi_3(f) = \frac{t_3 - t_1}{x} + 0 + \frac{t_3 - t_1}{t_2} + \frac{t_2 + x}{t_3}.$$

Suppose that $\prec$ is the purely lexicographic order on T_0, in which $t_1 \prec t_2 \prec \cdots \prec t_n$. Then $\prec$ is also a monomial order on each T_i, because $T_i \subseteq T_0$. For $f \in K_n$ and $i \in [n]_0$, the i-th projection of f can be viewed as a polynomial in $K_i[t_{i+1}, \ldots, t_n]$, which allows us to define the i*-th head monomial* of f, denoted by $\mathrm{hm}_i(f)$, to be the highest monomial in T_i that appears in $\pi_i(f)$ if $\pi_i(f)$ is non-zero, and zero if $\pi_i(f)$ is zero.

We define the i*-th head coefficient of* f, denoted by $\mathrm{hc}_i(f)$, to be the coefficient of $\mathrm{hm}_i(f)$ in $\pi_i(f)$ if $\pi_i(f)$ is non-zero, and zero if $\pi_i(f)$ is zero. By the matryoshka decomposition, $\mathrm{hc}_i(f)$ is t_i-proper for all $i \in [n]$.

The *head monomial* of f, denoted by $\mathrm{hm}(f)$, is defined to be the highest monomial among $\mathrm{hm}_0(f), \mathrm{hm}_1(f), \ldots, \mathrm{hm}_n(f)$, in which zero is regarded as the lowest "monomial". Let

$$I_f = \{i \in [n]_0 \mid \mathrm{hm}_i(f) = \mathrm{hm}(f)\}.$$

The *head coefficient* of f, $\mathrm{hc}(f)$, is defined to be $\sum_{i \in I_f} \mathrm{hc}_i(f)$.

Definition 3.2. *For $f, g \in K_n$, let d_f and d_g be the degrees of the denominators of f and g in t_n, respectively. We say that f is* lower than g, *denoted by $f \prec g$, if either $d_f < d_g$, or $d_f = d_g$ and* $\mathrm{hm}(f) \prec \mathrm{hm}(g)$. *We say that f is* not higher than g, *denoted by $f \preceq g$, if either $f \prec g$, or $d_f = d_g$ and* $\mathrm{hm}(f) = \mathrm{hm}(g)$.

For the rest of this article, we assume that $(K_0, ') = (C(x), d/dx)$ and each t_i in (4) is a primitive generator over K_{i-1} for all $i \in [n]$. Then we call K_n a *primitive extension* over K_0 and $K_0(\bar{t})$ a *primitive tower.* By Definition 2.1, $C_{K_n} = C_{K_0}$, which is equal to C. A primitive tower is said to be *logarithmic* if each t_i is a logarithmic generator over K_{i-1}.

Since $\prec$ on T_0 is a Noetherian total order, the partial order on K_n given by Definition 3.2 is also Noetherian, that is, every nonempty set in K_n has a minimal element with respect to $\prec$. We can use this order to define a desired remainder of the given function. Let $f \in K_n$ and

$$R_f := \{g \in K_n \mid g \equiv f \mod K_n'\}. \tag{6}$$

Thus, there exists a minimal element $r \in R_f$. We note that such a minimal element may not be unique. Furthermore, $\preceq$ is not a partial order, but rather a total preorder. Therefore, a minimal element of R_f with respect to $\prec$ is in fact a least element w.r.t. $\preceq$.

Definition 3.3. *Given $f \in K_n$, a minimal element of R_f is said to be a* remainder *of f. Moreover, let $r \in K_n$. Then we say that r is a remainder if r is a remainder of itself.*

As usual, t_i-simple elements play an important role when we construct remainders. Before we move on to the next section, we give a definition using the matryoshka decomposition.

Definition 3.4. *An element $f \in K_n$ is said to be* simple *if $\pi_i(f)$ is t_i-simple for all $i \in [n]_0$, where $t_0 = x$.*

Proposition 3.5. *Every logarithmic derivative in K_n is simple.*

Proof. We proceed by induction on n. Since every logarithmic derivative in K_0 is t_0-proper, the assertion holds for $n = 0$ by Lemma 2.6 (i). Assume that $n > 0$ and the assertion holds for $n - 1$. Let $f \in K_n$ be a logarithmic derivative. By Lemma 2.6 (ii), there exists a t_n-simple logarithmic derivative g and a logarithmic derivative $h \in K_{n-1}$ such that $f = g + h$. Then $\pi_n(f) = g$ by (2). Applying the induction hypothesis to h completes the induction. □

4 AN ADDITIVE DECOMPOSITION

Remainders in a tower are described in terms of minimality, which is not constructive. In this section, we will present an algorithm for constructing a remainder in an S-primitive tower based on Hermite reduction and integration by parts. To know when to terminate the algorithm, we need to be able to identify the first generator present in a given monomial (this is the same notion as *scale* in [6]).

Definition 4.1. *For a monomial* $M = t_1^{d_1} \cdots t_n^{d_n} \in T_0$*, the* indicator *of* M*, denoted by* $\operatorname{ind}_n(M)$*, is defined to be* n *if* $M = 1$*, or defined to be* $\min\{i \in [n] \mid d_i \neq 0\}$.

For $M \in T_0$, we set $K_n^{(\prec M)} := \{f \in K_n \mid \operatorname{hm}(f) \prec M\}$. Note that $K_n^{(\prec M)}$ is a C-linear subspace of K_n. The following lemma describes sufficient conditions for reducing a given term in a primitive tower with respect to $\prec$ via integration by parts.

Lemma 4.2. *Let* $K_0(\bar{t})$ *be a primitive tower, and* $M \in T_0$ *with indicator* m*. Then* $(f+ct_m)'M$ *belongs to* $K_n'+K_n^{(\prec M)}$ *for all* $f \in K_{m-1}$ *and* $c \in C$.

Proof. Let $M = t_m^{d_m} \cdots t_n^{d_n}$ for $d_m, \ldots, d_n \in \mathbb{N}$. Since K_n is a primitive extension over K_0, we see that $t_j' \in K_{j-1}$ for each j with $m \le j \le n$. Then $M' = \sum_{j=m}^{n} h_j N_j$, where $h_j \in K_{j-1}$, and

$$N_j = \begin{cases} t_j^{d_j-1} t_{j+1}^{d_{j+1}} \cdots t_n^{d_n} & d_j > 0, \\ 0 & d_j = 0. \end{cases}$$

So $f h_j \in K_{j-1}$ and $N_j \prec M$. Consequently, $f h_j N_j \prec M$ for all j with $m \le j \le n$. We see that $fM' \prec M$, which, together with $f'M = (fM)' - fM'$, implies that

$$f'M \in K_n' + K_n^{(\prec M)}. \tag{7}$$

It remains to show $t_m'M \in K_n'+K_n^{(\prec M)}$. Let $M = t_m^d N$, where $d \in \mathbb{N}$ and $N \in T_m$. Then $\operatorname{ind}_n(N) \ge m$ and

$$t_m'M = g'N, \tag{8}$$

where $g = t_m^{d+1}/(d+1)$. If $\operatorname{ind}_n(N) = m$, then $n = m$ and $N = 1$. Thus, $t_m'M \in K_n'$ by (8). Otherwise, $g'N \in K_n' + K_n^{(\prec N)}$ by (7), in which f and M are replaced with g and N, respectively. Moreover, $\operatorname{ind}_n(N) > m$ implies $N \prec M$. Thus, $t_m'M \in K_n'+K_n^{(\prec M)}$ by (8). ◻

In order to obtain sufficient and necessary conditions, we impose an extra condition on the generators:

$$\operatorname{hm}(t_i') = 1 \text{ for all } i \in [n].$$

By Lemma 2.5 and the additive decomposition for rational functions in $C(x)$, for each $i \in [n]$, there exists a simple element h_i in K_{i-1} and an element $g_i \in K_{i-1}$ such that $t_i' = g_i' + h_i$. Let $u_i = t_i - g_i$. Then u_i is a primitive generator over K_{i-1} and $K_{i-1}(t_i) = K_{i-1}(u_i)$. Moreover, $K_0(\bar{t}) = K_0(\bar{u})$. Therefore, without loss of generality, we can further assume that each t_i' is simple for all $i \in [n]$.

Definition 4.3. *A tower* $K_0(\bar{t})$ *is said to be* S-primitive *if it is a primitive tower and* t_i' *is simple for all* $i \in [n]$.

Logarithmic towers are S-primitive by Proposition 3.5. Our next goal is to construct remainders in S-primitive towers based on a special property of simple elements.

Lemma 4.4. *Let* $K_0(\bar{t})$ *be an S-primitive tower. If* $f \in K_n'$ *is simple, then* $f \in \operatorname{span}_C\{t_1', \ldots, t_n'\}$.

Proof. Since f is simple, $\pi_n(f)$ is t_n-simple, So $\pi_n(f) = \operatorname{hp}_{t_n}(f)$ by the uniqueness of Hermitian parts. Since $f \in K_n'$, we see that $\operatorname{hp}_{t_n}(f) = 0$ by Lemma 2.4 (i). Thus, $\pi_n(f) = 0$, and $f \in K_{n-1}$.

We proceed by induction on n. If $n = 1$, then $f \in K_0 \cap K_1'$ is x-simple by Definition 3.4. By Lemma 2.7, there exists a $c \in C$ such that $f \equiv ct_1' \mod K_0'$. Since both f and t_1' are x-simple, we have that $f = ct_1'$ by Lemma 2.4 (ii) and (iii). Assume that $n > 1$ and the lemma holds for $n-1$. For f in $K_{n-1} \cap K_n'$, there is a $c \in C$ such that $f \equiv ct_n' \mod K_{n-1}'$ by Lemma 2.7. Then $f - ct_n' \in K_{n-1}'$. Since both f and t_n' are simple, $f - ct_n'$ is also simple. By the induction hypothesis, we have that $f - ct_n' \in \operatorname{span}_C\{t_1', \ldots, t_{n-1}'\}$, which implies that $f \in \operatorname{span}_C\{t_1', \ldots, t_n'\}$. ◻

The previous lemma gives us a direct way to determine whether or not a tower is S-primitive.

Corollary 4.5. *The tower* $K_0(\bar{t})$ *is S-primitive if and only if* $t_1', \ldots, t_n'$ *are C-linearly independent and each* $t_i' \in K_{i-1}$ *is simple.*

Proof. If $K_0(\bar{t})$ is an S-primitive tower, then it is primitive. By Proposition 2.2, $t_i' \notin K_{i-1}'$ for all $i \in [n]$. So $t_1', \ldots, t_n'$ are C-linearly independent. By Definition 4.3, t_i' is simple for all $i \in [n]$.

We prove the converse by induction on n. If $n = 1$, then t_1' is nonzero because it is C-linearly independent, which implies $t_1' \notin K_0'$, because it is x-simple. By Proposition 2.2, t_1 is a primitive generator over K_0. Hence, $K_0(t_1)$ is S-primitive. Suppose that $K_0(t_1, \ldots, t_{n-1})$ is S-primitive. Let us consider the tower $K_0(t_1, \ldots, t_{n-1}, t_n)$. By Lemma 4.4, $t_n' \notin \operatorname{span}_C\{t_1', \ldots, t_{n-1}'\}$ implies that $t_n' \notin K_{n-1}'$. Thus, t_n is a primitive generator over K_{n-1} by Proposition 2.2. The tower under consideration is S-primitive. ◻

The following lemma gives a sufficient and necessary condition in S-primitive towers for lowering an element with respect to $\prec$ modulo the integrable space K_n'.

Lemma 4.6. *Suppose that* $K_0(\bar{t})$ *is an S-primitive tower. Let* $M \in T_0$ *with* $\operatorname{ind}_n(M) = m$ *and* $a \in K_{m-1}$ *be simple. Then* $aM \in K_n'+K_n^{(\prec M)}$ *if and only if* $a \in \operatorname{span}_C\{t_1', \ldots, t_m'\}$.

Proof. The sufficiency follows from Lemma 4.2. Conversely, assume that $aM \in K_n'+K_n^{(\prec M)}$. If $M = 1$, then $m = n$ and $a \in K_n'$. By Lemma 4.4, $a \in \operatorname{span}_C\{t_1', \ldots, t_n'\}$. Otherwise, since $\operatorname{ind}_n(M) = m$, assume that $M = t_m^{d_m} \cdots t_n^{d_n}$ with $d_m > 0$.

We proceed by induction on n. For $n = 1$, $aM \in K_1' + K_1^{(\prec M)}$ implies that there exists a t_1-proper element $b \in K_1$ and $p \in K_0[t_1]$ with $\deg_{t_1}(p) < d_1$ such that $aM + b + p \in K_1'$. We further assume that b is t_1-simple, because $b - \operatorname{hp}_{t_1}(b) \in K_1'$ by Lemma 2.5. So, $b = 0$ by Lemma 2.4 (i). We see that $aM + p \in K_1'$. By Lemma 2.7, $a - ct_1' \in K_0'$ for some $c \in C$. Hence, $a = ct_1'$, because a and t_1' are both x-simple.

Assume that $n > 1$ and that the conclusion holds for $n-1$. Let $N = M/t_n^{d_n}$, which is a power product of $t_m, \ldots, t_{n-1}$. Since aM belongs to $K_n' + K_n^{(\prec M)}$, there is a t_n-proper element b and $p \in K_{n-1}[t_n]$ with $\operatorname{hm}(p) \prec M$ such that $aNt_n^{d_n} + b + p \in K_n'$. Similar to the base case, one can show that $aNt_n^{d_n} + p \in K_n'$. Let

$p = qt_n^{d_n} + r$ such that $q \in K_{n-1}$ with $\mathrm{hm}(q) \prec N$ and $r \in K_{n-1}[t_n]$ with $\deg_{t_n}(r) < d_n$. Then we have $(aN + q)t_n^{d_n} + r \in K_n'$. By Lemma 2.7, there exists $c \in C$ such that $aN + q - ct_n' \in K_{n-1}'$. So,

$$aN \equiv ct_n' \mod \left(K_{n-1}' + K_{n-1}^{(\prec N)}\right). \tag{9}$$

If $N = 1$, then $m = n$ and $a \in K_n'$. So $a \in \mathrm{span}_C\{t_1', \ldots, t_n'\}$ by Lemma 4.4 and we are done. If $N \succ 1$, then $\mathrm{ind}_{n-1}(N) = m < n$. By (9), $aN \in K_{n-1}' + K_{n-1}^{(\prec N)}$, because $\mathrm{hm}(ct_n') = 1$. It follows from the induction hypothesis that $a \in \mathrm{span}_C\{t_1', \ldots, t_m'\}$. □

We can now specify a remainder in S-primitive towers and prove that the algorithm to construct it will terminate.

Proposition 4.7. *Let $K_0(\bar{t})$ be an S-primitive tower, and $r \in K_n$ be nonzero with $m = \mathrm{ind}_n(\mathrm{hm}(r))$. Then r is a remainder if $\pi_n(r)$ is t_n-simple, and* $\mathrm{hc}(r - \pi_n(r))$ *is simple and is not a nonzero element of* $\mathrm{span}_C\{t_1', \ldots, t_m'\}$.

Proof. Let $f \in R_r$ as defined in (6). Since $\pi_n(r)$ is t_n-simple, $\mathrm{hp}_{t_n}(f) = \pi_n(r)$ by Lemma 2.4 (ii) and (iii). Then the denominator of r, which is associated to the denominator of $\pi_n(r)$ over K_{n-1}, divides the denominator of f by Theorem 5.3.1 in [5].

We further need to show that $\mathrm{hm}(r) \preceq \mathrm{hm}(f)$. Suppose the contrary. Let $M = \mathrm{hm}(r)$ and $a = \mathrm{hc}(r - \pi_n(r))$.

If $M = 1$, then $m = n$, $a = r - \pi_n(r)$, and $f = 0$, which implies that $r \in K_n'$. Then $\pi_n(r) = 0$ by Lemma 2.4 (i). So, $a \in K_{n-1} \cap K_n'$. By Lemma 4.4, we have that a belongs to $\mathrm{span}_C\{t_1', \ldots, t_n'\}$. Thus, $a = 0$ and, consequently, $r = 0$, a contradiction.

Assume that $M \succ 1$. Then $\mathrm{hm}(r - f) = M$ and $\mathrm{hc}(r - f) = \mathrm{hc}(r)$ since $M \succ \mathrm{hm}(f)$. Hence, $\mathrm{hc}(r - f) = a$ because $M \succ 1$ and $\mathrm{hm}(\pi_n(r)) \preceq 1$. From $r - f \in K_n'$, we see that $a\,M \in K_n' + K_n^{(\prec M)}$. By Lemma 4.6, a belongs to $\mathrm{span}_C\{t_1', \ldots, t_m'\}$, which implies that $a = 0$. Then $r = \pi_n(r)$ and $M = 1$, a contradiction. □

Theorem 4.8. *Let $K_0(\bar{t})$ be an S-primitive tower and let $f \in K_n$. Then one can construct a remainder of f with the properties described in Proposition 4.7 in a finite number of steps.*

Proof. By Lemma 2.5, $\pi_n(f) \equiv \mathrm{hp}_{t_n}(f) \mod K_n'$. Then

$$f \equiv \mathrm{hp}_{t_n}(f) + (f - \pi_n(f)) \mod K_n'. \tag{10}$$

The n-th projection of the right-hand side of the congruence is equal to $\mathrm{hp}_{t_n}(f)$, which is t_n-simple.

Let $M = \mathrm{hm}(f - \pi_n(f))$. We proceed by a Noetherian induction on M with respect to $\prec$. If $M = 0$, then $f = \pi_n(f)$. By (10) and Proposition 4.7, $\mathrm{hp}_{t_n}(f) \in P_n$ is a remainder of f.

Assume that $M \neq 0$, and for any $g \in K_n$ with $\mathrm{hm}(g) \prec M$, there is a remainder $\tilde{r}$ of g as described in Proposition 4.7.

Let $a = \mathrm{hc}(f - \pi_n(f))$ and $m = \mathrm{ind}_n(M)$. Since $a \in K_{m-1}$, its j-th projection is equal to zero for each $j \in \{m, \ldots, n\}$. By Lemma 2.5, $\pi_i(a) \equiv h_i \mod K_i'$ for some t_i-simple elements $h_i \in K_i$ for all $i \in [m-1]_0$ with $t_0 = x$. By Lemma 4.2,

$$f - \pi_n(f) \equiv bM \mod (K_n' + K_n^{(\prec M)}), \tag{11}$$

where $b = \sum_{i=0}^{m-1} h_i$. Note that b is simple by Definition 3.4.

If $b \in \mathrm{span}_C\{t_1', \ldots, t_m'\}$, then $bM \in K_n' + K_n^{(\prec M)}$ by Lemma 4.2. So $f - \pi_n(f) \equiv g \mod K_n'$ for some g in $K_n^{(\prec M)}$ by (11). Accordingly, g has a remainder $\tilde{r}$ as described in Proposition 4.7 by the induction hypothesis. Thus, $\mathrm{hp}_{t_n}(f) + \tilde{r}$ is a remainder of f.

Assume that $b \notin \mathrm{span}_C\{t_1', \ldots, t_m'\}$. It follows from (10) and (11) that $f \equiv \mathrm{hp}_{t_n}(f) + bM + g \mod K_n'$ for some g in $K_n^{(\prec M)}$. We may further assume that $\pi_n(g)$ is t_n-simple by Lemma 2.5. The right-hand side of the above congruence is a remainder as described in Proposition 4.7, because b is the head coefficient of $bM + (g - \pi_n(g))$.

Consequently, we construct a remainder of f in a finite number of steps because the ordering $\prec$ is Noetherian. □

We now present an algorithm to decompose an element in an S-primitive tower into a sum of a derivative and a remainder. The algorithm is a slight refinement of the proof of the above theorem. We refer the reader to the online supplementary material [1] for the implementation.

AddDecompInField$\big(f,\ K_0(\bar{t})\big)$
Input: An S-primitive tower $K_0(\bar{t})$, described as a list

$$[x, [t_1, \ldots, t_n], [t_1', \ldots, t_n']],$$

s.t. $t_i' \in K_{i-1}$ is simple for all $i \in [n]$, and $f \in K_n$.
Output: Two elements $g, r \in K_n$ such that $f = g' + r$ and r satisfies the conditions in Proposition 4.7.

(1) If $f = 0$, then return $(0, 0)$.
(2) Initialize: $M \leftarrow \mathrm{hm}(f)$, $a \leftarrow \mathrm{hc}(f)$, $m \leftarrow \mathrm{ind}_n(M)$, $d \leftarrow \deg_{t_m}(M)$, $B \leftarrow 0$, $H \leftarrow 0$, $\tilde{c} \leftarrow 0$.
(3) Let $a = \sum_{i=0}^{m} a_i$ be the matryoshka decomposition.
(4) Reduction: For each i from 0 to m, compute $b_i, h_i \in K_i$ s.t. $a_i = b_i' + h_i$, where h_i is t_i-simple, and decide whether $\exists\, c_1, \ldots, c_m \in C$ s.t. $h_i = \sum_{j=1}^{m} c_j t_j'$.
  Yes: Update $B \leftarrow B + b_i + \sum_{j=1}^{m-1} c_j t_j$ and $\tilde{c} \leftarrow \tilde{c} + c_m$.
  No: Update $B \leftarrow B + b_i$ and $H \leftarrow H + h_i$.
(5) Lower term: $\ell \leftarrow f - aM - BM' - \frac{\tilde{c}}{d+1} \cdot t_m^{d+1} \cdot \left(M/t_m^d\right)'$.
  Recursion: $\{\tilde{g}, \tilde{r}\} \leftarrow$AddDecompInField$\big(\ell,\ K_0(\bar{t})\big)$.
(6) Return $g = BM + \frac{\tilde{c}}{d+1} \cdot t_m \cdot M + \tilde{g}$ and $r = H \cdot M + \tilde{r}$.

Example 4.9. *Find an additive decomposition for*

$$f = \frac{1}{\log(x)\mathrm{Li}(x)} + \frac{\mathrm{Li}(x) - 2x\log(x)}{(\log(x))^2} + \log(\log(x)),$$

viewed as an element of the S-primitive tower

$$K_3 = C(x)(\underbrace{\log(x)}_{t_1}, \underbrace{\mathrm{Li}(x)}_{t_2}, \underbrace{\log(\log(x))}_{t_3}),$$

and we can write $f = 1/(t_1 t_2) + (t_2 - 2xt_1)/t_1^2 + t_3 \in K_3$. By the above algorithm, we have that

$$f = \left(xt_3 + \frac{t_2^2}{2} - t_2 - \frac{xt_2 + x^2}{t_1}\right)' + \frac{1}{t_1 t_2}. \tag{12}$$

The nonzero remainder $r = 1/(t_1t_2)$ implies f has no integral in K_3.

An element $f \in K$ is said to have an *elementary integral over* K if there exists an elementary extension E of K and an element g of E such that $f = g'$ (see [5, Definition 5.1.4]). We can use the remainder from Theorem 4.8 to determine whether or not a function has an elementary integral over an S-primitive tower.

[1] https://wongey.github.io/add-decomp-sprimitive/

Theorem 4.10. *Let $K_0(\bar{t})$ be an S-primitive tower and C be algebraically closed. Let $f \in K_n$ have a remainder r as described in Proposition 4.7. Then f has an elementary integral over K_n if and only if $r \in \operatorname{span}_C\{t_1', \dots, t_n'\} + L_n$, where L_n stands for the C-linear subspace spanned by all logarithmic derivatives in K_n.*

Proof. The sufficiency is obvious. Conversely, there exists an $h \in L_n$ such that $f \equiv h \mod K_n'$ by Liouville's Theorem [5, Theorem 5.5.2]. Then it suffices to show that $r - h \in \operatorname{span}_C\{t_1', \dots, t_n'\}$.

Since r is a remainder of f, we have that $h \equiv r \mod K_n'$ and $\operatorname{hm}(r) \preceq \operatorname{hm}(h)$. By Proposition 3.5, h is simple, which implies that $\operatorname{hm}(h) \preceq 1$. So $\operatorname{hm}(r) \preceq 1$. If $\operatorname{hm}(r) = 0$, then $r = 0$. Otherwise, $\operatorname{hm}(r) = 1$. Then r is simple by Proposition 4.7. Thus, $r - h$ is simple and integrable in K_n. It is in $\operatorname{span}_C\{t_1', \dots, t_n'\}$ by Lemma 4.4. □

The proof of Theorem 4.10 gives us an alternate necessary condition, namely $\operatorname{hm}(r) \preceq 1$, to enable a quick check for the elementary integrability of f.

Example 4.11. *Let us reconsider the function f and the tower K_3 in Example 4.9 under the assumption that C is algebraically closed. The remainder is $r = t_2'/t_2$. By Theorem 4.10, f has an elementary integral over K_3. It follows from (12) that*

$$\int f\,dx = x\log(\log(x)) + \frac{\operatorname{Li}(x)^2}{2} - \operatorname{Li}(x) - \frac{x\operatorname{Li}(x) + x^2}{\log(x)} + \log(\operatorname{Li}(x)).$$

The Mathematica implementation by Raab based on work in [14] computes the same result. But the "int()" command in Maple and the "Integrate[ ]" command in Mathematica both leave the integral unevaluated.

As illustrated in Example 4.9, the function f therein has a nonzero remainder in $K_0(t_1, t_2, t_3)$. By Example 4.11, we see that zero is the remainder of f in $K_0(t_1, t_2, t_3)(t_4)$, where $t_4 = \log(t_2)$. However, to determine whether an element belongs to L_n given in Theorem 4.10, one needs the Rothstein-Trager resultant and algebraic numbers over C in general (see [5, Theorem 4.4.3] and [6, §6]), which may be complicated. We seek an easier way to find new generators.

5 LOGARITHMIC TOWERS

Let $K_0(\bar{t})$ and $K_0(\bar{u})$ be two primitive towers over K_0, and ϕ be a differential homomorphism from $K_0(\bar{t})$ to $K_0(\bar{u})$, which means ϕ is a field homomorphism and $\phi(f') = \phi(f)'$ for all $f \in K_0(\bar{t})$. For an element f of $K_0(\bar{t})$ with a remainder r, any remainder of $\phi(f)$ in $K_0(\bar{u})$ is always not higher than $\phi(r)$ with respect to $\prec$, because ϕ embeds the integrable subspace of $K_0(\bar{t})$ into that of $K_0(\bar{u})$.

In practice, determining generators for our towers depends heavily on the given function. In other words, the choice of generators can be done via a clever inspection of the function itself, as the following example shows.

Example 5.1. *Consider the following function in x:*

$$f = \frac{\log((x+1)\log(x))}{x\log(x)}.$$

For this function, there are at least two ways to construct the tower over $\mathbb{Q}(x)$ containing f:

(i) $t_1 = \log(x), t_2 = \log((x+1)\,t_1)$;
(ii) $u_1 = \log(x), u_2 = \log(x+1), u_3 = \log(u_1)$.

The tower $K_0(t_1, t_2)$ can be differentially embedded into $K_0(u_1, u_2, u_3)$ via $t_1 \mapsto u_1$ and $t_2 \mapsto u_2 + u_3$. In the first tower, $f = t_2/(xt_1)$ is already a remainder by Proposition 4.7. In the second tower, AddDecompInField *computes a remainder $u_2/(xu_1)$ that is lower than f because $\operatorname{span}_C\{t_1', t_2'\}$ is properly contained in $\operatorname{span}_C\{u_1', u_2', u_3'\}$.*

With the aid of the logarithmic derivative identity and the matryoshka decomposition, we are going to show in Theorem 5.6 that, given a logarithmic tower $K_0(t_1, \dots, t_n)$, one can construct another logarithmic tower $K_0(u_1, \dots, u_w)$ and a differential homomorphism ϕ such that, for all $i \in [n-1]_0, j \in [n]$, the image of $\pi_i(t_j')$ under ϕ belongs to $\operatorname{span}_C\{u_1', \dots, u_w'\}$, which provides us with more possibilities to reduce a given function by Lemma 4.6. This motivates the following representation of our towers in terms of the generators.

Definition 5.2. *Let $K_0(\bar{t})$ be a primitive tower. The $n \times n$ matrix*

$$A = \left(\pi_i(t_j')\right)_{0 \le i \le n-1, 1 \le j \le n}$$

is called the matrix associated *to $K_0(\bar{t})$.*

$$\begin{array}{cc} & \begin{array}{cccc} t_1' & t_2' & \cdots & t_n' \\ \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{array}{c} P_0 \rightarrow \\ P_1 \rightarrow \\ \vdots \\ P_{n-1} \rightarrow \end{array} & \left(\begin{array}{cccc} \star & \star & \cdots & \star \\ & \star & \cdots & \star \\ & & \ddots & \vdots \\ & & & \star \end{array}\right) \end{array}$$

Figure 3: A labeled associated matrix of a primitive tower. The $\star$ represents a possibly nonzero element.

The associated matrix records all information about the derivation on $K_0(\bar{t})$, because $\pi_n(t_1') = \cdots = \pi_n(t_n') = 0$. Since $t_j' \in K_{j-1}$ for all $j \in [n]$, the associated matrix A is in upper triangular form as in Figure 3. Furthermore, if $K_0(\bar{t})$ is a logarithmic tower, then the entries of A are all C-linear combinations of logarithmic derivatives by Lemma 2.6 (ii).

In the following discussion, a tower with a different set of generators $\bar{v} = (v_1, \dots, v_n)$ will appear. We say that $K_0(\bar{t})$ *is equal to* $K_0(\bar{v})$ if they are equal as a field, and that $K_0(\bar{t})$ is equal to $K_0(\bar{v})$ *as a tower* if $K_0(t_1, \dots, t_i) = K_0(v_1, \dots, v_i)$ for all $i \in [n]$. We will invoke the superscript notation to distinguish between different sets of generators (for example, $\pi_i^{\bar{t}}$ for projections in $K_0(\bar{t})$).

Definition 5.3. *Let $K_0(\bar{t})$ be a primitive tower and $f \in K_n \setminus \{0\}$. The* significant index *of f is*

$$\operatorname{si}^{\bar{t}}(f) := \max\{i \in [n]_0 \mid \pi_i(f) \neq 0\}.$$

The vector

$$\operatorname{sv}(\bar{t}) := \left(\operatorname{si}^{\bar{t}}(t_1'), \dots, \operatorname{si}^{\bar{t}}(t_n')\right)$$

is called the significant vector *of $K_0(\bar{t})$. Suppose $\operatorname{sv}(\bar{t})$ is equal to $(k_1, \dots, k_n)$. The sequence*

$$\operatorname{sc}(\bar{t}) := \left(\pi_{k_1}^{\bar{t}}(t_1'), \dots, \pi_{k_n}^{\bar{t}}(t_n')\right)$$

is called the significant component sequence *of $K_0(\bar{t})$.*

The significant vector and significant component sequence are unique with respect to the generators by the matryoshka decomposition.

Example 5.4. *Consider the field*

$$C(x)\,(\log(x), \log(\log(x)), \log((x+1)\log(x)))\,.$$

We set $t_1 = \log(x), t_2 = \log(t_1)$, *and* $t_3 = \log((x+1)\,t_1)$. *Then* $C(x)(t_1, t_2, t_3)$ *is a logarithmic tower whose significant vector is equal to* $(0, 1, 1)$ *and whose significant component sequence is*

$$(1/x, 1/(xt_1), 1/(xt_1)).$$

Definition 5.5. *A logarithmic tower* $K_0(\bar{t})$ *is said to be* well-generated *if*

(CLI) $\mathrm{sc}(\bar{t})$ *is* C*-linearly independent,*
(MI) $\mathrm{sv}(\bar{t})$ *is (weakly) monotonically increasing, and*
(ONE) *each column of its associated matrix contains exactly one non-zero element.*

$$\begin{pmatrix} \bullet & \cdots & \bullet & & & & & & & \\ & & & \bullet & \cdots & \bullet & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \bullet & \cdots & \bullet \\ & & & & & & & & & \end{pmatrix}$$

Figure 4: The associated matrix of a well-generated tower is in the form of a "staircase" where the •'s are C-linearly independent and other entries are zero.

We will show that any logarithmic tower $K_0(\bar{t})$ can be embedded into a well-generated one. To this end, we impose the usual lexicographical order on two significant vectors [10, Ch. 2, Def. 3].

Theorem 5.6. *Let* $K_0(\bar{t})$ *be a logarithmic tower. Then there exists a well-generated logarithmic tower* $K_0(\bar{u})$, *where* $\bar{u} = (u_1, \ldots, u_w)$ *and* $n \le w \le n(n+1)/2$, *and a differential monomorphism* ϕ *from* $K_0(\bar{t})$ *into* $K_0(\bar{u})$ *with* $\phi|_{K_0} = \mathrm{id}_{K_0}$.

Proof. This proof will be separated into two parts. The first part will show that each primitive (specifically, logarithmic) tower is equal (as a field) to one where properties (CLI) and (MI) are satisfied. This will enable us to embed the resulting logarithmic tower into a well-generated one, which makes up the second part of the proof.

If $K_0(\bar{t})$ does not satisfy (CLI) and (MI), then $\exists\, v_1, \ldots, v_n \in K_0(\bar{t})$ such that $K_0(\bar{v})$ is primitive and equals to $K_0(\bar{t})$, and $\mathrm{sv}(\bar{v})$ is lower than $\mathrm{sv}(\bar{t})$. Since the order of the significant vectors is Noetherian, we can eventually reach a primitive tower that satisfies both (CLI) and (MI).

We start by supposing that $\mathrm{sc}(\bar{t})$ is C-linearly dependent. Since all components of $\mathrm{sc}(\bar{t})$ are different from 0 by definition, there exists an $i \in \{2, \ldots, n\}$ and constants $c_1, \ldots, c_{i-1}$ such that

$$\mathrm{sc}_i = \sum_{j=1}^{i-1} c_j \mathrm{sc}_j,$$

where sc_j is the j-th element in $\mathrm{sc}(\bar{t})$. Moreover, $\mathrm{si}^{\bar{t}}(c_j t'_j) = \mathrm{si}^{\bar{t}}(t'_i)$ for all j with nonzero c_j. We remove the last non-zero projection of t'_i by setting $v_k := t_k$ for all $k \in [n] \setminus \{i\}$ and $v_i := t_i - \sum_{j=1}^{i-1} c_j t_j$. Thus, $K_0(\bar{v}) = K_0(\bar{t})$. Also, $\mathrm{si}^{\bar{v}}(v'_k) = \mathrm{si}^{\bar{t}}(t'_k)$ for all k in $[n] \setminus \{i\}$ and $\mathrm{si}^{\bar{v}}(v'_i) < \mathrm{si}^{\bar{t}}(t'_i)$. We conclude that $K_0(\bar{v})$ is a primitive tower with a lower significant vector than $K_0(\bar{t})$.

Next, we assume that $\mathrm{sv}(\bar{t})$ is not monotonically increasing. Then there exists an $i \in [n]$ such that $\mathrm{si}^{\bar{t}}(t'_1) \le \cdots \le \mathrm{si}^{\bar{t}}(t'_i)$ and $\mathrm{si}^{\bar{t}}(t'_{i+1}) < \mathrm{si}^{\bar{t}}(t'_i)$. We switch the i-th and $(i+1)$-st generators by setting $v_k := t_k$ for all $k \in [n] \setminus \{i, i+1\}$ and

$$v_i := t_{i+1};\ v_{i+1} := t_i.$$

Thus, $K_0(\bar{v})$ is equal to $K_0(\bar{t})$. Also, $\mathrm{si}^{\bar{v}}(v'_j) = \mathrm{si}^{\bar{t}}(t'_j)$ for $j \in [i-1]$ and $\mathrm{si}^{\bar{v}}(v'_i) < \mathrm{si}^{\bar{t}}(t'_i)$. Thus, $K_0(\bar{v})$ is a primitive tower with a lower significant vector than $K_0(\bar{t})$.

If the original primitive tower from the argument is logarithmic, then the new generators from the above process are also logarithmic generators. This implies the new tower must be logarithmic satisfying (CLI) and (MI), and this is what we assume about $K_0(\bar{t})$ from this point forward.

For the second part of the proof, we show that $K_0(\bar{t})$ can be embedded into a well-generated tower. We find the C-basis of the associated matrix $\left(\pi_i(t'_j)\right)$ by letting $b_1 = \pi_0(t'_1)$ and identifying all C-linearly independent elements $b_2, \ldots, b_w$, ordered by searching the matrix from left to right and top to bottom. Since $K_0(\bar{t})$ is primitive, $n \le w \le n(n+1)/2$. Since $K_0(\bar{t})$ satisfies (CLI) and (MI), there exist $\ell_1, \ldots, \ell_n \in [w]$ such that $\ell_1 = 1, \ell_n = w$,

$$\ell_1 < \ell_2 < \cdots < \ell_n \text{ and } \left(b_{\ell_1}, \ldots, b_{\ell_n}\right) = \mathrm{sc}(\bar{t}). \tag{13}$$

By the definition of the associated matrix and the ordering of $\{b_1, \ldots, b_w\}$, for all $j \in [n]$ there exist $c_{j,k} \in C$ such that

$$t'_j = b_{\ell_j} + \sum_{k=1}^{\ell_j - 1} c_{j,k} b_k. \tag{14}$$

Let $u_1, \ldots, u_w$ be algebraically independent indeterminates over K_0, and $\bar{u} := (u_1, \ldots, u_w)$. Let $v_j := u_{\ell_j} + \sum_{k=1}^{\ell_j - 1} c_{j,k} u_k$ for all $j \in [n]$. Then $v_1, \ldots, v_n$ are algebraically independent over K_0, because u_{ℓ_j} does not appear in the expressions defining $v_1, \ldots, v_{j-1}$. It follows that $\phi : K_0(\bar{t}) \to K_0(\bar{u})$ defined by $f(t_1, \ldots, t_n) \mapsto f(v_1, \ldots, v_n)$ is a monomorphism and $\phi|_{K_0} = \mathrm{id}_{K_0}$. For every $k \in [w]$, we define

$$u'_k = \phi(b_k). \tag{15}$$

Since $u_1, \ldots, u_w$ are algebraically independent over K_0, by Corollary 1′ in [17, page 124], the field $K_0(u_1, \ldots, u_w)$ can be uniquely turned into a differential field such that its derivation agrees with the one on K_0 and also satisfies (15). By (14), $\phi(t'_j) = v'_j$ for all $j \in [n]$. Thus, ϕ is a differential monomorphism.

Lastly, we show that $K_0(\bar{u})$ is a well-generated tower over K_0. Set $\ell_0 = 0$. For each $k \in [w]$, there exists a $j \in [n]$ such that $\ell_{j-1} < k \le \ell_j$. Then $s := \mathrm{si}^{\bar{t}}(b_k) \le \mathrm{si}^{\bar{t}}(t'_j) < j$ and b_k is t_s-proper. Since ϕ is a monomorphism, it preserves degrees. By (15), u'_k is u_{ℓ_s}-proper, where $\ell_s \le \ell_{j-1} < k$ since $s < j$. Hence, $u'_k \in K_0(u_1, \ldots, u_{k-1})$. Since ϕ is differential and b_k is a C-linear combination of logarithmic derivatives, so is u'_k by (15). In particular, u'_k is u_{ℓ_s}-simple by Lemma 2.6 (i). Moreover, $b_1, \ldots, b_w$ are C-linearly independent, and so are $\phi(b_1), \ldots, \phi(b_w)$ because ϕ is a monomorphism. It follows from (15) that $u'_1, \ldots, u'_w$ are C-linearly independent, which implies that $K_0(\bar{u})$ is a logarithmic tower by Corollary 4.5. In addition, $\pi_i(u_k') = 0$ for all $k \in [w]$ and $i \in [w] \setminus \{\ell_s\}$, because u_k' is u_{ℓ_s}-proper. Consequently, $K_0(\bar{u})$ is well-generated. □

The next example illustrates the results of the embedding algorithm and AddDecompInField in both towers.

Example 5.7. *Consider the logarithmic tower*

$$\mathcal{F} = C(x)\Big(\underbrace{\log(x)}_{t_1}, \underbrace{\log(xt_1)}_{t_2}, \underbrace{\log\left((x+1)(t_1+1)\log(xt_1)\right)}_{t_3}\Big).$$

By Theorem 5.6, there exists a well-generated tower

$$\mathcal{E} = C(x)\Big(\underbrace{\log(x)}_{u_1}, \underbrace{\log(x+1)}_{u_2}, \underbrace{\log(u_1)}_{u_3}, \underbrace{\log(u_1+1)}_{u_4}, \underbrace{\log(u_1+u_3)}_{u_5}\Big)$$

and a differential homomorphism ϕ from $\mathcal{F}$ to $\mathcal{E}$ given by $\phi(t_1) = u_1$, $\phi(t_2) = u_1 + u_3$ and $\phi(t_3) = u_2 + u_4 + u_5$. The associated matrices of $\mathcal{F}$ and $\mathcal{E}$ are, respectively,

$$\begin{pmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x+1} \\ 0 & \frac{t_1'}{t_1} & \frac{t_1'}{t_1+1} \\ 0 & 0 & \frac{1+t_1}{xt_1t_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{x} & \frac{1}{x+1} & 0 & 0 & 0 \\ 0 & 0 & \frac{u_1'}{u_1} & \frac{u_1'}{u_1+1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(u_1+u_3)'}{u_1+u_3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$f_1 = \frac{(t_1+1)^2 + t_1t_2}{xt_1(t_1+1)t_2} \quad \text{and} \quad f_2 = \frac{t_3}{x}$$

be two elements of $\mathcal{F}$. Then $\phi(f_1)$ and $\phi(f_2)$ are

$$\frac{(u_1+1)^2 + u_1(u_1+u_3)}{xu_1(u_1+1)(u_1+u_3)} \quad \text{and} \quad \frac{u_2+u_4+u_5}{x},$$

respectively. Using AddDecompInField, *we compute the respective remainders of f_1 and f_2 to obtain*

$$r_1 = f_1 \quad \text{and} \quad r_2 = \frac{t_1}{-(x+1)} + \frac{1}{x(t_1+1)} + \frac{-(t_1+1)}{xt_2}.$$

In the same vein, we get the remainders of $\phi(f_1)$ and $\phi(f_2)$,

$$\tilde{r}_1 = 0 \quad \text{and} \quad \tilde{r}_2 = \frac{u_1}{-(x+1)} + \frac{-(u_1+1)}{x(u_1+u_3)},$$

respectively. Note that $\phi(r_1) \neq 0$ but $\tilde{r}_1 = 0$, which implies that $\tilde{r}_1 \prec \phi(r_1)$. While $\mathrm{hm}(\tilde{r}_2) = \mathrm{hm}(\phi(r_2))$, we observe that $\tilde{r}_2$ has fewer nonzero projections than $\phi(r_2)$.

6 CONCLUSIONS

In this article, we have developed an additive decomposition in S-primitive towers. The decomposition algorithm is based on the matryoshka decomposition of functions, Hermite reduction and integration by parts. It provides an alternative method to Risch's algorithm for determining in-field (resp. elementary) integrability in (resp. over) an S-primitive tower. Moreover, we embed a logarithmic tower into a well-generated one, where functions can be decomposed further.

We observe that the notion of remainders is defined according to a partial order among multivariate rational functions. It would be possible to refine this notion so that all remainders of a given function share more common properties. Moreover, we plan to investigate whether our additive decomposition is applicable to compute telescopers for elements in an S-primitive tower, as carried out in [6]. We also hope to develop an additive decomposition in exponential extensions.

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