Differential Rational Normal Forms and a Reduction Algorithm for Hyperexponential Functions

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ABSTRACT

We describe differential rational normal forms of a rational function and their properties. Based on these normal forms, we present an algorithm which, given a hyperexponential function $T(x)$, constructs two hyperexponential functions $T_1(x)$ and $T_2(x)$ such that $T(x) = T_1'(x) + T_2(x)$ and $T_2(x)$ is minimal in some sense. The algorithm can be used to accelerate the differential Gosper's algorithm and to compute right factors of the telescopers.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

General Terms

Algorithms

Keywords

Normal forms; Rational functions; Hyperexponential functions; Reduction algorithms

1. INTRODUCTION

Let $\mathbb{F}$ be a field of characteristic 0. For a given rational function $R(x)$ over $\mathbb{F}$, any of the well-known reduction algorithms (see [5, Chap. 11] or [4, Chap. 2]) constructs two rational functions $R_1(x), R_2(x)$ in the field $\mathbb{F}(x)$ such that $R(x) = R_1'(x) + R_2(x)$ and the denominator of $R_2(x)$ has the minimal possible degree. (The symbol $'$ denotes the usual derivation w.r.t. x.)

A nonzero $T(x)$ is a hyperexponential function over $\mathbb{F}$, abbreviated hereafter as *h.e.f.*, if the ratio $T'(x)/T(x)$ is a rational function in $\mathbb{F}(x)$. This ratio is called the certificate of $T(x)$. For an h.e.f. $T(x)$, the differential Gosper's algorithm [3] determines whether there exists an h.e.f. $T_1(x)$ such that $T(x) = T_1'(x)$, and computes $T_1(x)$ provided that it exists.

Given an h.e.f. $T(x)$, we present a reduction algorithm which constructs two h.e.f.'s $T_1(x), T_2(x)$ such that $T(x)$ equals $(T_1'(x) + T_2(x))$, and $T_2(x)$ is minimal in some sense. The problem is defined so that it not only generalizes the reduction algorithms for rational functions, but also includes Gosper's algorithm as a special case, i.e., $T_2(x)$ is identically zero if $T(x)$ is hyperexponential integrable. This reduction algorithm avoids computing resultants and integer roots in Gosper's algorithm. This leads to an efficiency improvement (see Tables 1 and 2). The special structure of $T_2(x)$ allows us to define the notion of a *prescoper*, which is a right factor of the minimal telescoper of a bivariate h.e.f.

Inspired by the algorithm for solving the additive decomposition problem for hypergeometric terms [1], we propose a specific type of normal forms of rational functions. These normal forms and their construction are given in Sections 2 and 3. In Section 4, we discuss "similarity" among h.e.f.'s. In Section 5, we present a reduction algorithm for h.e.f.'s, and study properties of the output of the algorithm. This is the main section of the paper. Applications are presented in Section 6.

For $R \in \mathbb{F}(x)$, num(R) and den(R) denote the numerator and the denominator of R, respectively. Except when mentioned otherwise, num(R) and den(R) are co-prime, and den(R) is monic. The use of some technical terms is borrowed from [1]. The algorithms presented in this paper are implemented in the computer algebra system Maple, and are available from

`http://www.scg.uwaterloo.ca/~hqle/code/DRNF.html`.

2. DIFFERENTIAL NORMAL FORMS

In this section, we define differential rational normal forms (DRNF's) of a rational function $R(x)$. The construction is based on a classification and distribution of the simple fractions in the irreducible partial fraction decomposition of R. These DRNFs can be considered as the differential analogue of the RNFs in the difference case [1].

An ordered pair $(a, b) \in \mathbb{F}[x]\times\mathbb{F}[x]$ is said to be *differential-reduced* if $\gcd(b, a-ib')=1$ for all $i \in \mathbb{Z}$. A rational func-

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tion R in $\mathbb{F}(x)$ is differential-reduced if $(\text{num}(R), \text{den}(R))$ is differential-reduced (0 is evidently differential-reduced).

DEFINITION 1. *Let $R \in \mathbb{F}(x)$. If there are $K, S \in \mathbb{F}(x)$ such that (i) $R = K + S'/S$, (ii) K is differential-reduced, then (K, S) is a DRNF of R. We call K and S the* kernel *and the* shell *of the DRNF (K, S), respectively.*

A nonzero rational function R can be uniquely written as

$$R = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{q_{ij}}{d_i^j}, \tag{1}$$

where n, m_i are nonnegative integers, p, the d_i's and q_{ij}'s are in $\mathbb{F}[x]$, the d_i's are distinct, monic and irreducible, and q_{ij} is of degree less than that of d_i. We call (1) the irreducible partial fraction decomposition of R over $\mathbb{F}$. By a simple fraction we mean either a polynomial or a fraction whose denominator is a power of a square-free polynomial b of positive degree, and whose numerator is of degree less than the degree of b. An easy calculation shows

LEMMA 1. *A rational function is a logarithmic derivative of some element in $\mathbb{F}(x)$ iff its irreducible partial fraction decomposition can be written as $\sum_i n_i p_i'/p_i$ where the n_i's are nonzero integers and the p_i's are irreducible in $\mathbb{F}[x]$.*

The following lemma describes a relation between logarithmic derivatives and differential-reduced rational functions.

LEMMA 2. *A rational function R is differential-reduced iff, for any monic and irreducible p and nonzero integer m, the appearance of $(mp')/p$ in the irreducible partial fraction decomposition of R implies that p^2 divides $\text{den}(R)$.*

Proof: Set $a = \text{num}(R)$ and $b = \text{den}(R)$. Suppose that (a, b) is differential-reduced, and that mp'/p appears in the irreducible partial fraction decomposition, but that p^2 does not divide b. Then the irreducible partial fraction decomposition of a/b would be written as

$$\frac{a}{b} = \frac{u}{v} + \frac{mp'}{p} = \frac{up + mp'v}{vp}$$

where $u, v \in \mathbb{F}[x]$, $\gcd(u, v) = 1$, and $\gcd(p, v) = 1$. Since $\gcd(vp, up+mp'v) = 1$, we have $a = (up+mp'v)$ and $b = vp$. A direct calculation shows that p divides $\gcd(b, a - mb')$, a contradiction.

Conversely, suppose that (a, b) is not differential-reduced. It suffices to prove that there exist an irreducible p and a nonzero integer m such that (i) p^2 does not divide b, and (ii) mp'/p appears in the irreducible partial fraction decomposition of a/b. Since (a, b) is not differential-reduced, $g = \gcd(b, a - mb')$ is of positive degree for some nonzero integer m. Let p be an irreducible factor of g. Then the fact that $\gcd(a, b) = 1$ implies that p^2 does not divide b (hence, (i) is certified). It follows that $a/b = (u/v + q/p)$ where $\gcd(u, v) = 1$, $\gcd(v, p) = 1$ and $\deg q < \deg p$. Hence,

$$a = up + qv \quad \text{and} \quad b = vp.$$

As a consequence, $(a - mb') = (v(q - mp') + (u - mv')p)$. Since p divides $(a - mb')$, p divides $v(q - mp')$, and hence it divides $(q - mp')$. A degree argument implies $q = mp'$. Hence, mp'/p appears in the irreducible partial fraction decomposition of a/b. ∎

Consider the irreducible partial fraction decomposition

$$R = \sum_i \frac{u_i(x)}{v_i(x)}. \tag{2}$$

Each simple fraction u_i/v_i in (2) belongs to one of the following three classes: (I) $u_i/v_i = m_i\, v_i'/v_i,\ m_i \in \mathbb{Z}\setminus\{0\},\ v_i^2$ does not divide $\text{den}(R)$; (II) $u_i/v_i = m_i\, v_i'/v_i,\ m_i \in \mathbb{Z} \setminus \{0\},\ v_i^2$ divides $\text{den}(R)$; (III) u_i/v_i is not a logarithmic derivative of any rational function.

Let (K, S) be a DRNF of R. Then the simple fractions in class (I) appear in the irreducible partial fraction decomposition of S'/S, not in the irreducible partial fraction decomposition of K (otherwise, K is not differential-reduced); the simple fractions in class (III) appear in the irreducible partial fraction decomposition of K, not in the irreducible partial fraction decomposition of S'/S (otherwise, S'/S would not be a logarithmic derivative of any rational function); the simple fractions in class (II) can appear in the irreducible partial fraction decomposition of either K or S'/S.

LEMMA 3. *Let u_i/v_i be a simple fraction of class (I) or (II) in (2), then v_i is irreducible.*

Proof: If v_i is not irreducible, then $v_i = p^s$ where p is irreducible and $s > 1$. Because u_i/v_i is of class (I) or (II), by definition, $u_i/v_i = m_i\, s\, p'/p$. Since $\deg p < \deg v_i$, $\gcd(u_i, v_i)$ is not trivial, a contradiction. ∎

The following corollary follows from Lemma 3.

COROLLARY 1. *Let the irreducible partial fraction decomposition of nonzero $R \in \mathbb{F}(x)$ be of the form (2). For $i \neq j$, if the simple fraction u_i/v_i is of class (I) and u_j/v_j is of either class (II) or class (III), then $\gcd(v_i, v_j) = 1$.*

LEMMA 4. *The denominator of the kernel of a DRNF is unique.*

Proof: In (2), each simple fraction u_i/v_i of class (II) is of the form

$$\frac{u_i}{v_i} = m_i\,\frac{v_i'}{v_i},\ m_i \in \mathbb{Z} \setminus \{0\},\ v_i^2 \text{ divides } \text{den}(R).$$

Since v_i^2 divides $\text{den}(R)$ and the denominators of the simple fractions of classes (I) and (II) are pairwise co-prime (Corollary 1), there is a simple fraction of class (III) such that its denominator can be written as v_i^s, $s \geq 2$. Hence, the denominator of the kernel of any DRNF is the denominator of the sum of the simple fractions in class (III). ∎

EXAMPLE 1. Consider the rational function

$$\begin{aligned} R &= \frac{4}{x-2} + \frac{4}{x+1} - \frac{3}{(x+1)^2} - \frac{9}{(x-1)^2} \\ &\quad - \frac{9x^2+12}{x^3+4x-2} + \frac{1}{(x^3+4x-2)^2}. \end{aligned}$$

The simple fractions of R are classified as follows:

$$\text{(I)}\ \ u_1 = \frac{4}{x-2}, \qquad \text{(II)}\ \ v_1 = \frac{4}{x+1},\ v_2 = -\frac{9x^2+12}{x^3+4x-2},$$

$$\text{(III)}\ w_1 = -\frac{9}{(x-1)^2}, w_2 = -\frac{3}{(x+1)^2}, w_3 = \frac{1}{(x^3+4x-2)^2}.$$

Now we construct four different DRNF's of R.

The first DRNF is constructed by moving both simple fractions in class (II) to the shell:

$$\left(w_1 + w_2 + w_3,\ \frac{\operatorname{den}(u_1)^4 \operatorname{den}(v_1)^4}{\operatorname{den}(v_2)^3}\right).$$

The second DRNF is constructed by moving both simple fractions in class (II) to the kernel:

$$\left(w_1 + w_2 + w_3 + v_1 + v_2,\ \operatorname{den}(u_1)^4\right).$$

The third DRNF is constructed by moving v_1 to the shell and v_2 to the kernel:

$$\left(w_1 + w_2 + w_3 + v_2,\ \operatorname{den}(u_1)^4 \operatorname{den}(v_1)^4\right).$$

Finally, the fourth DRNF is constructed by moving v_2 to the shell and v_1 to the kernel:

$$\left(w_1 + w_2 + w_3 + v_1,\ \frac{\operatorname{den}(u_1)^4}{\operatorname{den}(v_2)^3}\right).$$

3. A DIFFERENTIAL CANONICAL FORM

Among all possible DRNFs of a rational function $R(x)$, we select one canonical form (DRCF) whose kernel K is the sum of the simple fractions of classes (II) and (III) in (2), and whose shell is the rational function S such that S'/S is the sum of the simple fractions of class (I). By Lemma 2, the kernel K is differential-reduced. Note that the DRCF of a rational function also appears in [7, Chap. 8].

For the rational function $R(x)$ in Example 1, the second DRNF is the DRCF of R. The next theorem shows the minimality of the shell of the DRCF.

THEOREM 1. *For $R \in \mathbb{F}(x)$, let S be the shell of the DRCF of R, and $\tilde{S}$ be the shell of any DRNF of R. Then* $\operatorname{den}(S)$ *divides* $\operatorname{den}(\tilde{S})$, *and* $\operatorname{num}(S)$ *divides* $\operatorname{num}(\tilde{S})$.

Proof: Let R be of the form (2), A and B be the sets of simple fractions of class (I) and class (II), respectively. Each element f of either A or B is of the form $m\,v'/v$ where v is monic, and irreducible in $\mathbb{F}[x]$, and m, denoted by $\operatorname{res}(f)$, is a nonzero integer. Then

$$S = \prod_{f \in A} \operatorname{den}(f)^{\operatorname{res}(f)},\ \ \tilde{S} = S \underbrace{\prod_{g \in J} \operatorname{den}(g)^{\operatorname{res}(g)}}_{W} \qquad (3)$$

where J is a subset of B. By Corollary 1, $\operatorname{den}(f)$ and $\operatorname{den}(g)$ are co-prime. Hence,

$$\operatorname{num}(\tilde{S}) = \operatorname{num}(S)\operatorname{num}(W),\ \operatorname{den}(\tilde{S}) = \operatorname{den}(S)\operatorname{den}(W). \quad \blacksquare$$

Corollary 1 and the first equality of (3) imply if (K, S) is the DRCF of a rational function, $\operatorname{den}(K)$, $\operatorname{num}(S)$ and $\operatorname{den}(S)$ are pairwise co-prime.

Although the DRCF of $R \in \mathbb{F}(x)$ can be directly read off from the full irreducible partial fraction decomposition (1), we can construct the DRCF without computing (1). Let S be the shell of the DRCF of R, and the sum $\sum_{i=1}^{k} m_i p_i'/p_i$ be the irreducible partial fraction decomposition of S'/S, where the m_i's are nonzero integers. Let f be the product of irreducible factors of $\operatorname{den}(R)$ with multiplicity one. By the definition of DRCF's, f is divisible by each of the p_i's. Write $R = p + g/f + v/u$, where $p, f, g, u, v \in \mathbb{F}[x]$ with $\operatorname{den}(R)$=fu, $\deg g < \deg f$ and $\deg v < \deg u$. It follows that the fraction $m_i p_i'/p_i$ appears in the irreducible partial fraction decomposition of S iff it appears in that of g/f. Consequently, a monic irreducible factor q of f is equal to one of the p_i's iff there exists a nonzero integer m such that q does not divide the denominator of the difference of g/f and mq'/q. In other words, the numerator $(g - mq'w)$ of this difference, where $w = f/q$, is divisible by q. This constraint gives rise to a system of linear equations in a single unknown. Such an integer m exists iff it is the integral solution of the system. In practice, this method for constructing S is less time-consuming than that by computing (1).

4. SIMILARITY

Let $T(x)$ be an h.e.f. with the certificate $R \in \mathbb{F}(x)$. Let the pair of rational functions (K, S) be a DRNF of R. Then T can be written in the form $T(x) = S(x)\exp\left(\int K(x)\,dx\right)$. Such a form is called a *multiplicative decomposition* of T.

Two h.e.f.'s T_1 and T_2 are similar if their ratio can be written as the product of a rational function and a constant in some extension of $\mathbb{F}$, or equivalently, the difference between the rational certificates of T_1 and T_2 is a logarithmic derivative of a rational function. Similarity is an equivalence relation. If $T(x)$ is an h.e.f., then $T'(x)$ is an h.e.f. similar to $T(x)$. Let $T_1(x)$ and $T_2(x)$ be hyperexponential such that $T_1(x) + T_2(x) \neq 0$. Then $T_1(x) + T_2(x)$ is hyperexponential iff $T_1(x)$ is similar to $T_2(x)$. The next lemma shows the use of DRNFs in determining the similarity of two h.e.f.'s.

LEMMA 5. *Let $T_1(x), T_2(x)$ be two h.e.f.'s, and (K_1, S_1), (K_2, S_2) be DRNFs of the certificates of T_1 and T_2, respectively. If T_1 and T_2 are similar, then* $\operatorname{den}(K_1) = \operatorname{den}(K_2)$.

Proof: Since T_1 and T_2 are similar,

$$K_1 - K_2 = Q'/Q \quad \text{for some nonzero } Q \in \mathbb{F}(x). \qquad (4)$$

Let p be an irreducible factor of $\operatorname{den}(K_1)$ with multiplicity m. Then a simple fraction q/p^m must appear in the irreducible partial fraction decomposition of K_1. If $m > 1$, then there is a simple fraction q/p^m appearing in the irreducible partial fraction decomposition of K_2, because all simple fractions in the irreducible partial fraction decomposition of Q'/Q have square-free denominators by Lemma 1. If $m = 1$, then $q \neq ip'$ for any integer i, for, otherwise, K_1 is not differential-reduced by Lemma 2. It follows from (4) that there exists a simple fraction f/p in the irreducible partial fraction decomposition of K_2 such that the difference of q/p and f/p is a logarithmic derivative of some rational function. Therefore p^m is also a factor of $\operatorname{den}(K_2)$. Consequently, $\operatorname{den}(K_1)$ divides $\operatorname{den}(K_2)$. In the same way, $\operatorname{den}(K_2)$ divides $\operatorname{den}(K_1)$. $\blacksquare$

5. A REDUCTION ALGORITHM

5.1 Algorithm description

An h.e.f. $T(x)$ over $\mathbb{F}$ is said to be hyperexponential integrable if there exists an h.e.f. T_1 such that $T = T_1'$. The reduction problem for h.e.f.'s can be specified as follows.

Given an h.e.f. T, find an h.e.f. T_1 and a function T_2, which is either zero or an h.e.f. such that $T = T_1' + T_2$ and

(i) if T is hyperexponential integrable, then $T_2 = 0$,

(ii) if T is not hyperexponential integrable, then T_2'/T_2 has a DRNF (K, S) such that the denominator of S has the minimal possible degree.

This formulation agrees with that of the reduction algorithms for rational functions [4, 5] since if $T_2 \in \mathbb{F}(x)$ then $\mathrm{num}(K) = 0$, $\mathrm{den}(K) = 1$, and $\mathrm{den}(S) = \mathrm{den}(T_2)$. An easy calculation shows

LEMMA 6. *Let T be an h.e.f. If there are h.e.f.'s T_1, T_2 such that $T = T_1' + T_2$, then T, T_1 and T_2 are pairwise similar. If (K, S) and (K, S_1) are respective multiplicative decompositions of T and T_1, then the pair (K, S_2) is a multiplicative decomposition of T_2 where $S_2 = S - S_1' - S_1 K$.*

The following lemma, which is the core of Hermite's reduction algorithm for rational functions [5, page 484] will play an essential role in our proposed algorithm.

LEMMA 7. *Let $B(x) = r(x)/q(x)^j$ be a simple fraction with $j>1$ and $\deg q > 0$. Then there are $e, f \in \mathbb{F}[x]$,*

$$\deg e < (\deg q) - 1,\ \deg f < \deg q \tag{5}$$

such that

$$B(x) = \left(\frac{-f(x)/(j-1)}{q(x)^{j-1}}\right)' + \frac{e(x) + f'(x)/(j-1)}{q(x)^{j-1}}. \tag{6}$$

For a rational function $u_1(x)/u_2(x)$, $u_1, u_2 \in \mathbb{F}[x]$, let the square-free factorization of u_2 be $\prod_{i=1}^{k} q_i^i$. Then

$$\frac{u_1(x)}{u_2(x)} = p + \sum_{i=1}^{k}\sum_{j=1}^{i} \frac{r_{ij}}{q_i^j}, \tag{7}$$

where for $1 \le i \le k$ and $1 \le j \le i$, $p, r_{ij} \in \mathbb{F}[x]$ and

$$\deg r_{ij} < \deg q_i \text{ if } \deg q_i > 0, \text{ and } r_{ij} = 0 \text{ if } q_i = 1. \tag{8}$$

The main idea of our algorithm is contained in the following theorem and its proof.

THEOREM 2. *(Hermite-like reduction) Let R be a nonzero rational function with the DRCF (K, S). Write $K = k_1/k_2$ where $k_1, k_2 \in \mathbb{F}[x]$. Then there are $S_1 \in \mathbb{F}(x)$, $u_1, u_2 \in \mathbb{F}[x]$ such that (i) $S - S_1' - S_1 K = u_1/(u_2 k_2^i)$, $i \in \{0, 1\}$, (ii) u_2 is square-free, (iii) $\gcd(k_2, u_2) = 1$.*

Proof: Let $B = r_{ij}/q_i^j$ be a simple fraction of S such that j is maximal and greater than one. Then

$$S = \frac{a}{s_2} + \frac{r_{ij}}{q_i^j}, \quad \text{where } a, s_2 \in \mathbb{F}[x] \text{ and } q_i^j \nmid s_2. \tag{9}$$

Apply Lemma 7 to B to obtain $e, f \in \mathbb{F}[x]$ such that relations (5) and (6) hold. Set $S_{1,1} = -\frac{f/(j-1)}{q_i^{j-1}}$. Then it follows from Lemma 7 that

$$S - S_{1,1}' - S_{1,1} K = \frac{a}{s_2} + \frac{e + f'/(j-1)}{q_i^{j-1}} + \frac{f/(j-1)}{q_i^{j-1}}\frac{k_1}{k_2}. \tag{10}$$

Since (K, S) is the DRCF of R (which implies $\mathrm{den}(S)$ and $\mathrm{den}(K)$ are co-prime), and since q_i divides $\mathrm{den}(S)$, we have $\gcd(q_i, k_2) = 1$. Hence, the left hand side of (10) can be written as $c_0/t_2 + c_1/k_2 + c_2/q_i^m$, where $t_2, c_0, c_1, c_2 \in \mathbb{F}[x]$, $\deg c_2 < \deg q_i$, and $0 \le m < j$. In addition, q_i^m does not divide t_2 if $m > 0$ and $c_2 = 0$ if $m = 0$. Repeating this step if necessary on c_2/q_i^m and on the simple fractions of c_0/t_2 of the form (7) by using $S_{1,2}, S_{1,3}, \ldots$, we obtain

$$S - (S_{1,1}' + S_{1,2}' + \cdots) - (S_{1,1} + S_{1,2} + \cdots) K \tag{11}$$

whose denominator is of the form $u_2 k_2^i$, $i \in \{0, 1\}$, where u_2 is square-free. The rational function $(S_{1,1} + S_{1,2} + \cdots)$, and the numerator of (11) are the required rational function S_1, and the numerator u_1, respectively. Since $\gcd(k_2, s_2) = 1$ and $u_2 \mid s_2$, we have $\gcd(k_2, u_2) = 1$. ■

Let (K, S) be a multiplicative decomposition of an h.e.f. T. Lemma 6 and Theorem 2 allow one to construct two similar h.e.f.'s $T_1(x), T_2(x)$ with multiplicative decompositions (K, S_1) and (K, S_2), respectively, where $S_2 = u_1/(u_2 k_2^i)$, $i \in \{0, 1\}$, u_1, u_2, S_1 are as defined in the proof of Theorem 2 such that $T(x) = T_1'(x) + T_2(x)$. If k_2 exists in the denominator of S_2, one can rewrite $T_2(x)$ in a simpler form by removing the factor k_2 in the denominator of S_2:

$$T_2 \quad = \quad \frac{u_1}{u_2} \exp\left(\int (k_1 - k_2')/k_2 \, dx\right).$$

It is easy to check that the rational function $(k_1 - k_2')/k_2$ is differential-reduced. This leads to the following theorem.

THEOREM 3. *Let T be an h.e.f. Then there exists an h.e.f. T_1 similar to T such that the difference $(T - T')$ is either zero or an h.e.f. whose the rational certificate has a DRNF (K, S) which satisfies the following two properties: (i) $\mathrm{den}(S)$ is square-free, and (ii) $\gcd(\mathrm{den}(K), \mathrm{den}(S)) = 1$.*

DEFINITION 2. *A pair of rational functions (K, S) is* indecomposable *if K is differential-reduced, $\mathrm{den}(S)$ is square-free, and $\gcd(\mathrm{den}(K), \mathrm{den}(S)) = 1$.*

An algorithmic description of Theorem 2 is

`Algorithm` **ReduceCert**
`input:` $D, U \in \mathbb{F}(x)$ where (D, U) is the DRCF of some $R \in \mathbb{F}(x)$;
`output:` $U_1, K, S \in \mathbb{F}(x)$ such that
1. $K + S'/S = D + U_2'/U_2$, $U_2 = U - U_1' - U_1 D$,
2. (K, S) is indecomposable.

$U_1 := 0;\ U_2 := U;\ u_2 := \mathrm{den}(U_2);$
let $U = \sum_{i=1}^{k}\sum_{j=1}^{i} r_{ij}/q_i^j$ be the square-free decomposition of U;
for i from 1 to k do
  for j from i downto 2 do
    if $q_i^j \mid u_2$ then
      write $U_2 = a/\tilde{u_2} + b/q_i^j$, $a, b, \tilde{u_2} \in \mathbb{F}[x]$;
      apply Lemma 7 to $R = b/q_i^j$ to compute $e, f \in \mathbb{F}[x]$ such that (5), (6) hold;
      $\tilde{U_1} := -(f/(j-1))/q_i^{j-1}$;
      $U_2 := U_2 - \tilde{U_1}' - \tilde{U_1} D$;
      $U_1 := U_1 + \tilde{U_1}$; $u_2 := \mathrm{den}(U_2)$;
    fi;
  od;
od;
$k_1 := \mathrm{num}(D)$; $k_2 := \mathrm{den}(D)$; $s_1 := \mathrm{num}(U_2)$; $s_2 := u_2$;
if $k_2 \mid s_2$ then
  $s_2 := s_2/k_2$; $k_1 := k_1 - k_2'$;
fi;
return $(U_1, k_1/k_2, s_1/s_2)$.

EXAMPLE 2. *For* $T = \dfrac{1}{(x-1)^2 x^3} \exp\left(\displaystyle\int \frac{2x-7}{(x+4)^2} dx\right)$,

$$(D, U) = \mathrm{DRCF}(T'/T) = \left(\frac{2x-7}{(x+4)^2}, \frac{1}{(x-1)^2 x^3}\right).$$

Applying algorithm ReduceCert to (D,U) results in a triple of rational functions (U_1,K,S) which equals

$$\left(-\frac{89x^2-41x-16}{32(x-1)x^2}, -\frac{15}{(x+4)^2}, \frac{89x^2-1424x-1225}{32(x-1)x}\right).$$

Hence, $T(x)$ is decomposed into two similar h.e.f.'s $T_1(x)$ and $T_2(x)$ such that $T(x)=T_1'(x)+T_2(x)$ where

$$T_1 = -\frac{89x^2-41x-16}{32(x-1)x^2}\exp\left(\int \frac{2x-7}{(x+4)^2}dx\right),$$
$$T_2 = \frac{89x^2-1424x-1225}{32(x-1)x}\exp\left(\int -\frac{15}{(x+4)^2}dx\right).$$

5.2 Integrability

Let T be hyperexponential integrable. Applying the reduction algorithm in Section 5.1 to T yields

$$T = U' + \underbrace{S\exp\left(\int K\right)}_{H} \qquad (12)$$

where U is hyperexponential and the pair (K,S) is indecomposable. What special properties does (K,S) satisfy ? The following theorem provides a partial answer.

THEOREM 4. *Let T and T_1 be h.e.f.'s such that $T=T_1'$. Let (K,S) and (K,S_1) be multiplicative decompositions of T and T_1, respectively. If the pair (K,S) is indecomposable, then both S and S_1 are polynomials.*

Proof: Let $s_1=\text{num}(S)$, $s_2=\text{den}(S)$, $k_1=\text{num}(K)$ and $k_2=\text{den}(K)$. The assumption

$$\frac{s_1}{s_2}\exp\left(\int\frac{k_1}{k_2}\right) = \left(S_1\exp\left(\int\frac{k_1}{k_2}\right)\right)' \text{ implies that}$$

$$S_1' + \frac{k_1}{k_2}S_1 = \frac{s_1}{s_2}. \qquad (13)$$

First, we show that $s_2=1$. Suppose the contrary. Then s_2 has a root r in some algebraic extension of $\mathbb{F}$. Since s_2 is square-free, the order of s_1/s_2 at r is one. Since s_2 and k_2 are co-prime, the order of K at r is zero. Let m be the order of S_1 at r. If $m>0$, then the order of the left hand-side of (13) is equal to $m+1$ which is greater than 1. If $m=0$, then the order of the left-hand side of (13) equals 0 which is less than 1. Hence, $\deg s_2 = 1$, i.e., $S\in\mathbb{F}[x]$.

The proof that S_1 is a polynomial is a variant of that of the theorem in [3, page 577]. Rewrite (13) as $S_1'+\frac{k_1}{k_2}S_1=s_1$. Let $S_1=a/b$, a, $b\in\mathbb{F}[x]$, $\gcd(a,b)=1$. Then

$$\frac{a'b-ab'}{b^2}+\frac{k_1}{k_2}\frac{a}{b}=s_1. \qquad (14)$$

Clearing denominators of (14) yields

$$k_2\,a'\,b - k_2\,a\,b' + k_1\,a\,b = k_2\,s_1\,b^2. \qquad (15)$$

Suppose that $\deg b\geq 1$. For some $h\in\mathbb{N}\setminus\{0\}$, let

$$b = A^h\,\bar b,\ \deg A\geq 1,\ \gcd(A,\bar b)=1,\ \gcd(A,A')=1. \qquad (16)$$

It follows from (15) and (16) that

$$A\left(k_2\,a'\,\bar b - k_2\,a\,\bar b' + k_1\,a\bar b - k_2\,s_1\,A^h\,\bar b^2\right) = k_2\,a\,h\,A'\bar b. \qquad (17)$$

Since $\gcd(A, A'\,\bar b\,a)=1$, (17) implies that $A\,|\,k_2$. Write

$$k_2 = A\,\bar k_2,\ \bar k_2\in\mathbb{F}[x]. \qquad (18)$$

It follows from (17) and (18) that

$$A\left(\bar k_2\,a'\,\bar b - \bar k_2\,a\,\bar b' - \bar k_2\,s_1\,A^h\,\bar b^2\right) = -a\,\bar b\left(k_1 - h\,\bar k_2\,A'\right).$$

Since A does not divide $(-a\,\bar b)$, $A\,|\,(k_1-h\,\bar k_2\,A')$. Additionally, $A\,|\,(-h\,A\,\bar k_2')$. Hence, $A\,|\,(k_1-h\,k_2')$ by (18), a contradiction since $A\,|\,k_2$ and (k_1,k_2) is differential-reduced. ∎

In order to decide if T is hyperexponential integrable, we only need to decide if H is so according to (12). By Theorem 4, we may conclude that H is not hyperexponential integrable if S is not a polynomial in $\mathbb{F}[x]$; otherwise, we need to find a polynomial solution S_1 of the equation (13). If such a solution S_1 exists, then $\text{den}(K)\,|\,S_1$, because S is a polynomial. Hence, we compute a polynomial f such that

$$\text{den}(K)f' + (\text{den}(K)' + \text{num}(K))f = S \qquad (19)$$

and set $S_1 = f\text{den}(K)$. Note that (19) is of the same form as (G8) in [3]. So the special techniques developed in [3] can be directly applied to finding polynomial solutions of (19).

The combination of ReduceCert and Theorem 4 allows one to design an algorithm which solves the reduction problem for h.e.f's as specified at the beginning of Section 5.1.

`Algorithm` **ReduceHyperexp**
`input:` an h.e.f. T;
`output:` two h.e.f.'s T_1, T_2 such that $T=T_1'+T_2$ and
(i) if T is hyperexponential integrable, T_2=0;
(ii) otherwise, T_2'/T_2 has a DRNF (K,S) with $\text{den}(S)$ of minimal degree;

$(D,U) := DRCF(T'/T)$;
$(U_1,K,S) := ReduceCert(D,U)$;
if $\deg\text{den}(S)>0$ then
  return $\left(U_1\exp\left(\int D\right), S\exp\left(\int K\right)\right)$;
else
  if (19) has a polynomial solution f then
    return $\left(U_1\exp\left(\int D\right) + f\text{den}(K)\exp\left(\int K\right), 0\right)$;
  else return $\left(U_1\exp\left(\int D\right), S\exp\left(\int K\right)\right)$;
  fi;
fi.

EXAMPLE 3. *Applying algorithm ReduceHyperexp to*

$$T = \frac{2x^4-x^3+x^2-2x-1}{x^2(x+1)^2}\exp\left(\int\frac{x}{(x+1)^2}dx\right)$$

yields a triple of rational functions (U_1,K,S):

$$\left(\frac{1}{2x}, -\frac{3x+4}{(x+1)^2}, x^4+\frac{3}{2}x^3+\frac{1}{2}x^2-\frac{1}{2}x-1\right).$$

Since S is a polynomial, and since the equation (19) admits $f=\frac{(x^3-x^2-4x-1)}{2}$ as a polynomial solution, T is hyperexponential integrable, and $T=T_1'$ where

$$T_1 = \frac{x^4-x^3-4x^2-x+1}{2x}\exp\left(\int -\frac{3x+4}{(x+1)^2}dx\right).$$

5.3 Algorithm verification

We verify that the algorithm ReduceHyperexp solves the reduction problem specified at the beginning of Section 5.1. The output of the algorithm has property (i) by Theorem 4, and has property (ii) by the next theorem, which also has applications in Section 6.2.

Theorem 5. *Let the h.e.f.'s T, T_1, $\tilde{T}_1$ be such that*

$$T_2 = T - T_1', \ \tilde{T}_2 = T - \tilde{T}_1'.$$

Let (K,S) and $(\tilde{K},\tilde{S})$ be multiplicative decompositions of T_2 and $\tilde{T}_2$, respectively, where both K and $\tilde{K}$ are differential-reduced. If (K,S) is indecomposable, then $\mathrm{den}(\tilde{S})$ is divisible by $\mathrm{den}(S)$.

Proof: Since T_2 and $\tilde{T}_2$ are similar by Lemma 6, Lemma 5 implies that $\mathrm{den}(K) = \mathrm{den}(\tilde{K})$, which is denoted by f. The similarity between T_2 and $\tilde{T}_2$ implies that $(\tilde{K} - K)$ is equal to r'/r for some $r \in \mathbb{F}(x)$. Hence $\mathrm{den}(r'/r)$ is a factor of f. One can verify that $\mathrm{den}(r'/r)$ is the product of the square-free parts of $\mathrm{num}(r)$ and $\mathrm{den}(r)$. Consequently, both $\mathrm{num}(r)$ and $\mathrm{den}(r)$ are factors of some power of f. Hence

$$\exp\left(\int \tilde{K}\right) = \frac{h}{f^k}\exp\left(\int K\right) \tag{20}$$

where $h \in \mathbb{F}[x]$. Write $S = s_1/s_2$ and $\tilde{S} = \tilde{s}_1/\tilde{s}_2$, where $s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \mathbb{F}[x]$, $\gcd(s_1, s_2) = 1$ and $\gcd(\tilde{s}_1, \tilde{s}_2) = 1$.

We construct a suitable multiplicative decomposition of $\tilde{T}_2$ so that Theorem 2 is applicable. Write $\tilde{s}_2 = uw$ where $\gcd(u,f) = 1$ and w is a factor of f^m for some nonnegative integer m. Then the fraction $\frac{\tilde{s}_1}{\tilde{s}_2} = \frac{e}{uf^m}$, where e is a polynomial co-prime to u, and

$$\tilde{T}_2 = \frac{e}{u}\exp\left(\int \underbrace{\left(\tilde{K} - \frac{mf'}{f}\right)}_{G_m}\right). \tag{21}$$

Note that u is a factor of $\tilde{s}_2$, G_m is differential-reduced and $\gcd(u,f) = 1$. Apply ReduceHyperexp to the right-hand side of (21) to get

$$\tilde{T}_2 = T_3' + \frac{w_1}{w_2 f^n}\exp\left(\int \tilde{K}\right) \tag{22}$$

where T_3 is an h.e.f., $w_1, w_2 \in \mathbb{F}[x]$, $\gcd(w_1, w_2) = 1$, w_2 is square-free, $\gcd(w_2, f) = 1$ and n is either m or $(m+1)$. The reduction process implies that w_2 is a divisor of the square-free part of u. So $w_2 \,|\, \tilde{s}_2$. Therefore it suffices to show $s_2 \,|\, w_2$.

Note that $\tilde{T}_2 = T_3' + \frac{w_1 h}{w_2 f^{n+k}}\exp\left(\int K\right)$ by (20) and (22). Since $(T_2 - \tilde{T}_2)$ is a derivative of some h.e.f.,

$$G = \left(\frac{s_1}{s_2} - \frac{w_1 h}{w_2 f^{n+k}}\right)\exp\left(\int K\right)$$

is a derivative of some h.e.f. Now we construct a multiplicative decomposition of G so that Theorem 4 is applicable.

Let $g = \gcd(s_2, w_2)$ and write $s_2 = pg$ and $w_2 = qg$. Then

$$G = \frac{s_1 f^{n+k} q - w_1 h p}{pqg}\exp\left(\int \underbrace{\left(K - \frac{(n+k)f'}{f}\right)}_{G_{n+k}}\right).$$

Since G_{n+k} is differential-reduced, pqg is square-free, and $\gcd(f, pqg) = 1$, by Theorem 4, $(s_1 f^{n+k} q - w_1 h p)/(pqg)$ is a polynomial. Hence $p \,|\, (s_1 f^{n+k} q - w_1 h p)$, and consequently $p \,|\, (s_1 f^{n+k} q)$. Since $\gcd(p, s_1 f q) = 1$, $\deg p = 0$. Hence, $s_2 = g$ which is a factor of w_2. As $w_2 \,|\, \tilde{s}_2$, $s_2 \,|\, \tilde{s}_2$. ∎

6. APPLICATIONS

6.1 Differential Gosper's algorithm

In this section we compare the differential Gosper's algorithm in [3] and our reduction algorithm ReduceHyperexp, both are implemented in Maple 9 [1] (see also the function contgosper in [8]).

The differential Gosper's algorithm constructs the equation (G8), which is of the same form as (19), by three calculations: calculating Sylvester's resultant of $\mathrm{den}(R)$ and $(\mathrm{num}(R) - z\,\mathrm{den}(R)')$, where R is the certificate of T and z is an indeterminate, finding the nonnegative integer roots of the resultant (regarded as a polynomial in z), and computing gcd's of polynomials. The reduction algorithm also constructs (19) by three calculations: finding the DRCF (K,S) of R, computing the square-free partial fraction decomposition of S, and performing a Hermite-like reduction on the partial fractions of S. We applied these two algorithms to h.e.f's generated in various ways.

In the first suite, the rational function R_j is the ratio of randomly generated polynomials r_{1j} and r_{2j}, where the range of the integral coefficients of r_{ij} is from $-100{,}000$ to $+100{,}000$, and $\deg r_{1j} = j$, $\deg r_{2j} = j+1$. We applied the two algorithms to $T_j = \exp\left(\int R_j\,dx\right)$. Usually, the shell of the DRCF of R_j is equal to one. So the three calculations to obtain (19) in the reduction algorithm are often trivial. However, Gosper's algorithm still needs to calculate the resultant to get the equation (G8). Consequently, the reduction algorithm is much faster than Gosper's algorithm. Note that T_j is usually not hyperexponential-integrable. Next, Gosper's algorithm and the reduction algorithm are applied to T_j', which is hyperexponential-integrable. The rational certificate of T_j' usually has the DRCF $(R_j,\, R_j)$. So the first calculation in the reduction algorithm is nontrivial. However, the last two are often trivial because the shell R_j has a square-free denominator. Twelve sets of tests were used. Table 1 shows the average time requirement for the input T_j' ($\mathcal{G}$ is for Gosper's algorithm, and $\mathcal{R}$ is for the reduction algorithm.)

Table 1: Average time requirement of $\mathcal{G}$ and $\mathcal{R}$

Timing (seconds)					
j	$\mathcal{G}$	$\mathcal{R}$	j	$\mathcal{G}$	$\mathcal{R}$
50	3.233	0.410	110	43.299	2.621
60	5.734	0.606	120	60.545	3.263
70	9.144	1.060	130	80.459	4.091
80	14.384	1.319	140	106.264	4.761
90	21.917	1.725	150	138.043	6.173
100	31.661	2.197	160	174.174	6.899

In the second suite, the rational function Q_j is equal to $R_j + S_j'/S_j$, where R_j, S_j are rational functions generated in the same way as in the first suite. The shell of the DRCF of Q_j is usually equal to S_j and $T_j = \exp\left(\int Q_j dx\right)$ is usually not hyperexponential-integrable. Since $\mathrm{den}(S_j)$ is often square-free, the last two calculations in the reduction algorithm take little time, and the third rational function in the output of ReduceCert often has a nontrivial denominator.

[1] All the reported timings were obtained on a 1Ghz Compaq Deskpro Workstation with 512Mb RAM.

Theorem 4 then tells us that T_j is not hyperexponential-integrable. Hence, we do not need to compute the polynomial solutions of (19). On the other hand, Gosper's algorithm would have to perform all the three calculations and to compute the polynomial solutions of (G8). Empirical data shows that the reduction algorithm is much more efficient than Gosper's algorithm for h.e.f's of this kind. Next, Gosper's algorithm and the reduction algorithm are applied to T'_j, which is hyperexponential-integrable. The rational certificate of T'_j usually has a DRNF $(R_j, S_jR_j + S'_j)$. So the three calculations in the reduction algorithm are all non-trivial. Twelve sets of tests were used. Table 2 shows the average time requirement for the input T'_j.

Table 2: Average time requirement of G and R

Timing (seconds)					
j	$\mathcal{G}$	$\mathcal{R}$	j	$\mathcal{G}$	$\mathcal{R}$
50	23.300	15.535	110	334.726	157.582
60	41.053	23.778	120	453.496	210.189
70	72.146	40.537	130	619.422	294.178
80	107.800	59.946	140	815.346	370.196
90	160.642	83.393	150	1047.920	463.027
100	237.471	117.254	160	1319.165	584.567

6.2 Minimal prescopers

In this section, unless otherwise mentioned, by an h.e.f. T we mean an h.e.f. in both x and y, i.e., both the x-certificate $\partial_x T/T$ and y-certificate $\partial_y T/T$ belong to $\mathbb{F}(x,y)$. Recall that a telescoper of T is a nonzero linear differential operator L in $\mathbb{F}(x)[\partial_x]$ such that the application $L(T)$ of L to T is hyperexponential integrable w.r.t. y. A telescoper is minimal if it is of minimal degree in ∂_x. We shall show that the reduction algorithm ReduceHyperexp in Section 5.2 may help us factor minimal telescopers.

Following [2], define a pair $(P, Q) \in \mathbb{F}(x,y) \times \mathbb{F}(x,y)$ to be *differentially compatible* if $\partial_y P = \partial_x Q$. For $R \in \mathbb{F}(x,y)$, $\mathrm{den}(R)$ and $\mathrm{num}(R)$ denote the denominator and numerator of R, respectively. They belong to $\mathbb{F}[x,y]$ and are co-prime. The next lemma describes a relation between the denominators of P and Q.

Lemma 8. *If P and Q in $\mathbb{F}(x,y)$ are differentially compatible, then $\mathrm{den}(P)/\mathrm{den}(Q) = f(x)/g(y)$ for some $f(x)$ in $\mathbb{F}[x]$, and $g(y)$ in $\mathbb{F}[y]$.*

Proof: Let $\mathrm{den}(P) = a^m b$ where a is square-free, and a, b are co-prime, $\mathrm{num}(P) = c$, $\mathrm{den}(Q) = a^k u$ with $\gcd(a,u) = 1$, and $\mathrm{num}(Q) = v$. Then

$$\partial_y P = \frac{(\partial_y c)ab - mcb(\partial_y a) - ca(\partial_y b)}{a^{m+1}b^2}, \tag{23}$$

$$\partial_x Q = \frac{(\partial_x v)au - kvu(\partial_x a) - va(\partial_x u)}{a^{k+1}u^2}. \tag{24}$$

Since a and $cb(\partial_y a)$ are co-prime, a^{m+1} divides $\mathrm{den}(\partial_y P)$. Hence, a^{m+1} divides a^{k+1} by $\partial_y P = \partial_x Q$ and (24), and consequently, a^m divides $\mathrm{den}(Q)$.

Suppose further that $\deg_x a > 0$ and that k is the multiplicity of a in Q. Switching the roles of P and Q, we find $m \geq k$ by $\partial_y P = \partial_x Q$ and (23). The factor a has the same multiplicities in both P and Q if $\deg_x a$ and $\deg_y a$ are positive. Write

$$\mathrm{den}(P) = f_1(x)g_1(y)h_1(x,y), \ \mathrm{den}(Q) = f_2(x)g_2(y)h_2(x,y),$$

where $f_i \in \mathbb{F}[x]$, $g_i \in \mathbb{F}[y]$ and $h_i \in \mathbb{F}[x,y]$ whose contents w.r.t. x and w.r.t. y are trivial. The conclusions reached in the last two paragraphs imply $g_1|g_2$, $f_2|f_1$ and $h_1 = h_2$. ∎

An h.e.f. can be expressed as

$$T(x,y) = \exp\left(\int P(x,y)\,dx + Q(x,y)\,dy\right), \tag{25}$$

where the pair $(P,Q) \in \mathbb{F}(x,y) \times \mathbb{F}(x,y)$ are differentially compatible. In fact, P and Q are the x- and y-certificates of T, respectively. Two h.e.f.'s with the same certificates can differ by a multiplicative constant. The reduction algorithm is applicable to an h.e.f. w.r.t. y when we use the following rule to modify integrands: for $R \in \mathbb{F}(x,y)$ and T in (25),

$$RT = \exp\left(\int \left(P + \frac{\partial_x R}{R}\right) dx + \left(Q + \frac{\partial_y R}{R}\right) dy\right). \tag{26}$$

This rule keeps the certificates differentially compatible.

Let (K, S) be the DRCF w.r.t y of Q in (25). By (26),

$$T = S \underbrace{\exp\left(\int \left(P - \frac{\partial_x S}{S}\right) dx + K\,dy\right)}_{H}.$$

By Theorem 2 there are $R \in \mathbb{F}(x,y)$, $u, v \in \mathbb{F}(x)[y]$ with u being square-free, $\gcd(u,v) = 1$ and $\gcd(u, \mathrm{den}(K)) = 1$, such that $S - \partial_y R - RK = \frac{v}{u\,\mathrm{den}(K)^i}$, where $i \in \{0,1\}$. Hence, we have

$$T = \partial_y(T_1) + \frac{v}{u} T_2, \tag{27}$$

where $T_1 = RH$, $T_2 = H$ if $i = 0$, and T_2 equals

$$\exp\left(\int \left(P - \frac{\partial_x(S\,\mathrm{den}(K))}{S\,\mathrm{den}(K)}\right) dx + \left(K - \frac{\partial_y \mathrm{den}(K)}{\mathrm{den}(K)}\right) dy\right),$$

if $i = 1$. By Theorem 4, T is not hyperexponential integrable w.r.t. y if $\deg_y u > 0$. This observation motivates us to define the notion of prescopers.

Definition 3. *A differential operator $L \in \mathbb{F}(x)[\partial_x]$ is called a* prescoper *of an h.e.f. T w.r.t. y if $L(T)$ can be written as a sum $\partial_y T_1 + pT_2$ where T_1, and T_2 are h.e.f.'s, the y-certificate of T_2 is differential-reduced w.r.t. y, and p belongs to $\mathbb{F}(x)[y]$. A nonzero prescoper of minimal degree in ∂_x is called a* minimal prescoper.

Clearly, a telescoper is a prescoper.

We define a sequence of mappings $\mathcal{R}_i$ from $\mathbb{F}(x,y)$ to itself recursively. Let $\mathcal{R}_0$ send everything to one, and $\mathcal{R}_i$ send an element $r \in \mathbb{F}(x,y)$ to $\partial_x(\mathcal{R}_{i-1}(r)) + r\mathcal{R}_{i-1}(r)$ for $i \in \mathbb{Z}^+$. An easy induction shows that $\partial_x^i(T)$ in (25) equals $\mathcal{R}_i(P)T$.

Lemma 9. *Let L belong to $\mathbb{F}(x)[\partial_x]$, T be an h.e.f. given in (25), and r in $\mathbb{F}(x,y)$. Then $L(rT)$ equals aT where a is in $\mathbb{F}(x,y)$ and $\mathrm{den}(a)$ is a factor of the product of some power of $\mathrm{den}(r)$ and some power of $\mathrm{den}(Q)$ over $\mathbb{F}(x)$.*

Proof: Observe that $L(rT)$ is an $\mathbb{F}(x)$-linear combination of the products of $\partial_x^i r$, $\mathcal{R}_j(P)$ and T, where $i,j \in \mathbb{N}$. Hence, the lemma holds since $\mathrm{den}(P) = \mathrm{den}(Q)/g$ for some g in $\mathbb{F}(x)[y]$ by Lemma 8. ∎

The following proposition implies that minimal prescopers are right factors of minimal telescopers.

PROPOSITION 1. *Let T be an h.e.f. and I_T be the set of prescopers of T. Then I_T is a left ideal of $\mathbb{F}(x)[\partial_x]$. In particular, the minimal prescoper of T is a right factor of the minimal telescoper of T.*

Proof: Let L be a prescoper of T. Then

$$L(T) = \partial_y H + p\,G \tag{28}$$

where H, G are h.e.f.'s, the y-certificate of G is differential-reduced w.r.t. y, and p belongs to $\mathbb{F}(x)[y]$. For any element $M \in \mathbb{F}(x)[\partial_x]$, we need to show that ML is in I_L. Denote by Q the y-certificate of G. By Lemma 9 and the commutativity of ∂_y with any element of $\mathbb{F}(x)[\partial_x]$, applying M to (28) yields $ML(T) = \partial_y(M(H)) + qG$, where q is in $\mathbb{F}(x,y)$ and $\mathrm{den}(q)$ is a factor of $\mathrm{den}(Q)^m$ for some nonnegative integer m. Thus, for some $r \in \mathbb{F}(x)[y]$,

$$ML(T) = \partial_y(M(H)) + \frac{r}{\mathrm{den}(Q)^m}G = \partial_y(M(H)) + r\tilde{G},$$

where $\tilde{G}$ is an h.e.f. with differential-reduced y-certificate. The operator ML is a prescoper.

It is a more involved to show that I_T is closed under addition, because we need to specify constants more explicitly when adding up two similar h.e.f.'s. Let L_1 and L_2 be differential operators in I_T. By Definition 3 we get

$$L_1(T) = \partial_y(T_1) + p_1 H_1 \text{ and } L_2(T) = \partial_y(T_2) + p_2 H_2, \tag{29}$$

where T_1, T_2, H_1, H_2 are h.e.f.'s, H_1 and H_2 have differential-reduced y-certificates, and p_1, p_2 belong to $\mathbb{F}(x)[y]$. Since H_1 and H_2 are similar when treated as h.e.f.'s in y, their y-certificates have the same denominator, say p, by Lemma 5, and the denominator of their ratio H_1/H_2 is a factor of p^k for some nonnegative integer k by (20). There then exist a constant c w.r.t. y and a polynomial q in $\mathbb{F}(x)[y]$ such that $H_1 = cqH_2/p^k$. So the equalities in (29) imply

$$(L_1 + L_2)(T) = (\partial_y H) + \frac{f}{p^k}H_2 = (\partial_y H) + f\tilde{H}_2, \tag{30}$$

where f is in $\mathbb{F}(c,x)[y]$, H and $\tilde{H}_2$ are h.e.f.'s w.r.t. y, and the y-certificate of $\tilde{H}_2$ is differential-reduced. Now, applying the reduction algorithm to $(L_1 + L_2)(T)$, we get

$$(L_1 + L_2)(T) = (\partial_y G) + uH_3 \tag{31}$$

where G and H_3 are h.e.f's, and u is in $\mathbb{F}(x,y)$. However, u has to be a polynomial by (30) and Theorem 5. It follows from (31) that $(L_1 + L_2)$ is a prescoper.

Since I_L is a left ideal of $\mathbb{F}(x)[\partial_x]$, it is principal with the minimal prescoper as the generator, which is a right factor of any element of I_L. ■

Due to the page limitations, we merely outline an idea on constructing the minimal prescoper M of an h.e.f. T. Apply the reduction algorithm to T to get (27). Clearly, M is also the minimal prescoper of $H = \frac{v}{u}T_2$ given in (27). If $\deg_y u$ is zero, M is equal to 1 and we gain nothing. If $\deg_y u$ is positive, M is nontrivial by Theorem 5. For $m = 1, 2, \ldots$, we apply the differential operator

$$L_m = \partial_x^m + a_{m-1}\partial_x^{m-1} + \cdots + a_0,$$

where the a_i's are unspecified functions in $\mathbb{F}(x)$, to T. Apply the reduction algorithm to $L_m(T)$ to get

$$L_m(T) = \partial_y(T_1) + \frac{t}{s}T_2,$$

where T_1, T_2 are h.e.f's, s belongs to $\mathbb{F}(x)[y]$ and t belongs to $\mathbb{F}(x)[y, a_0, \ldots, a_{m-1}]$. Note that the a_i's appear linearly in t. The assumption that $s \mid t$ over $\mathbb{F}(x)$ results in a linear system S_m in $a_{m-1}, \ldots, a_0$. The first consistent S_m gives M. The existence of prescopers implies that we will reach an integer m with the consistent S_m.

EXAMPLE 4. *Compute the minimal prescoper and telescoper of rT, where $T = \exp\left(\frac{x^2-y^2}{(y-1)^2}\right)$, $r = \left(\frac{1}{(y-x^2)x^2} - \frac{1}{yx^2}\right)$. the reduction algorithm returns $(T_1, T_2) = (1, rT)$. The minimal prescoper M of rT is:*

$$\partial_x^2 + \frac{19x^2-2+11x^6-24x^4-2x^8}{x(5x^2-4x^4-2+x^6)}\partial_x - \frac{4(x^6-4x^4+2x^2+2)}{x^2(x^4-3x^2+2)}.$$

Since the minimal telescoper of $M(rT)$ is

$$L = \partial_x + \frac{x^3(5x^2-7)}{5x^2-4x^4-2+x^6},$$

the minimal telescoper of rT is equal to LM.

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