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Factoring systems of linear PDEs with finite-dimensional solution spaces

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Abstract

A D-finite system is a finite set of linear homogeneous partial differential equations in several independent and dependent variables, whose solution space is of finite dimension. Let L be a D-finite system with rational function coefficients. We present an algorithm for computing all hyperexponential solutions of L, and an algorithm for computing all D-finite systems whose coefficients are also rational functions, and whose solutions are contained in the solution space of L.

1. Introduction

For various reasons *linear* differential equations have been of particular importance in the history of mathematics. First of all, the problems connected with them are much easier than those for nonlinear equations. Second, many nonlinear problems may be linearized in some way such that the results of the former theory may be applied to them. This is especially true for Lie's symmetry analysis of ordinary differential equations (ODEs) which reduces the problem of solving nonlinear ODEs with a sufficiently large number of symmetries to the study of certain systems of linear partial differential equations (PDE's). The problem of finding conservation laws for nonlinear PDE's also leads to systems of linear PDE's.

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Many concepts from commutative algebra have been suitably generalized to the algebraic theory of linear ODEs, e.g. the greatest common divisor and least common multiple, the concept of reducibility and factorization which finally led to the theory of Picard and Vessiot and differential Galois theory. This is true to a much lesser extent for linear homogeneous PDE's. To obtain manageable problems, we have to specialize them further. The constraint that the general solution depends on a finite number of constants, i.e. it may be represented as a linear combination with constant coefficients of a finite number of special solutions which form a fundamental system, turns out to be appropriate. It allows us to generalize many concepts from the theory of linear ODEs in an almost straightforward manner. Linear homogeneous PDE's with this constraint will be called *D-finite systems*, which may be seen as a slight generalization of Definition 2.1 given by Chyzak and Salvy (1998) in the differential case. They arise from different research areas such as: symmetry analysis (Lie, 1873), holonomic systems (Saito et al., 2000) and the description of functions by a given system of PDE's with initial conditions (Chyzak and Salvy, 1998). We shall focus on D-finite systems whose coefficients are rational functions in the independent variables over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$.

Contributions of this paper mainly consist of two algorithms. The first finds all hyperexponential solutions of a D-finite system. It generalizes the main algorithm by Li and Schwarz (2001) to the case of several independent variables. The second finds all D-finite systems whose coefficients are in the field of rational functions over $\overline{\mathbb{Q}}$, and whose solution spaces are properly contained in the solution space of a given system. It generalizes the factorization algorithm sketched by Tsarev (2001) and completed by Li et al. (2002) to the case of several independent and dependent variables. In principle, most problems related to D-finite systems reduce, as shown by Lie, to corresponding problems for linear ODEs. However, such "reduction" may be nontrivial and usually leads to solving or factoring linear ODEs with parameters. This makes many known algorithms fail. We shall avoid such complications. The paper also proves a theorem describing the structure of hyperexponential solutions of a D-finite system in one dependent variable (Proposition 3.4), and generalizes the notion of left quotients of linear ODEs to D-finite systems (Proposition 5.3).

This paper is based on several known results. The theory of linear differential ideals (Kolchin, 1973) supplies useful conclusions about dimension and linear dependence. The reduction–completion process (Janet, 1920; Galligo, 1985; Kandru-Rody and Weispfenning, 1990; Schwarz, 1992; Chyzak and Salvy, 1998) makes sure that the systems to be factored and the factors to be sought are of required rank. The idea of associated equations (Beke, 1894; Schlesinger, 1895; Schwarz, 1989; Bronstein, 1994) inspires us to reduce our factorization problem to that of finding hyperexponential solutions of associated systems.

The paper is organized as follows. Section 2 contains necessary preliminaries. Section 3 presents an algorithm for computing all hyperexponential solutions of a D-finite system in one dependent variable. Section 4 extends the results of Section 3 to the case of several dependent variables. Section 5 presents a factorization algorithm. Some concluding remarks are given in Section 6.

2. Preliminaries

We shall specify notation, state problems to be studied, and list a few useful results in this section.

All (sub)modules, vector spaces and ideals in the paper are left (sub)modules, left vector spaces and left ideals, respectively.

Throughout this paper, the following notation will be used: the symbol $\mathbb{K}$ stands for the field $\overline{\mathbb{Q}}(x_1, \ldots, x_n)$. The field $\mathbb{K}$ is viewed as a partial differential field on which usual derivation operators $\partial_1 = \partial/\partial x_1, \ldots, \partial_n = \partial/\partial x_n$ act. Denote by Θ the commutative monoid generated by $\partial_1, \ldots, \partial_n$, and by $\mathbb{K}_i$ the field $\overline{\mathbb{Q}}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $i = 1, \ldots, n$.

The symbol $\mathbb{D}$ stands for the noncommutative ring $\mathbb{K}[\partial_1, \ldots, \partial_n]$ of differential operators (see Chyzak and Salvy, 1998; Saito et al., 2000 or van der Put and Singer, 2003, Appendix D). For f in $\mathbb{D}$ and a in a differential field extension of $\mathbb{K}$, we denote the application of f to a by $f(a)$.

Let $y_1, \ldots, y_m$ be m differential indeterminates over $\mathbb{K}$. For every θ in Θ, θy_i is called a *derivative*. The set of all derivatives is denoted by Γ. We denote by $\mathbb{L}_m$ the $\mathbb{K}$-linear space generated by all elements of Γ. Then $\mathbb{L}_m$ is both a $\mathbb{K}$-linear space and a module over $\mathbb{D}$. An element f of $\mathbb{L}_m$ can be written as

$$f = \sum_{\gamma \in \Gamma} f_\gamma \gamma, \tag{1}$$

where $f_\gamma \in \mathbb{K}$, only finitely many nonzero. Alternatively, we may regard $\mathbb{L}_m$ as the sum $\sum_{i=1}^{n} \mathbb{D} y_i$ of left modules over $\mathbb{D}$, where $\mathbb{D} y_i = \{f(y_i) \mid f \in \mathbb{D}\}$, that is, $\mathbb{L}_m$ is isomorphic to the direct sum of m copies of $\mathbb{D}$ as left modules over $\mathbb{D}$. We opt for the notation $\mathbb{L}_m$, because the introduction of unknowns $y_1, \ldots y_m$ makes it easier to speak about solutions of differential equations.

A subset L of $\mathbb{L}_m$ is called a submodule if $f(a) \in L$ for every $f \in \mathbb{D}$ and $a \in L$. For a subset S of $\mathbb{L}_m$ the submodule generated by S, denoted by (S), is the linear space spanned by the elements of $\{f(s) \mid f \in \mathbb{D}, s \in S\}$. Every submodule of $\mathbb{L}_m$ is finitely generated by Proposition 1.9 of Chapter V in Borel et al. (1987) or the basis theorem on p. 126 in Kolchin (1973). A submodule L is said to be of *finite rank* if the quotient $(\mathbb{L}_m/L)$ is a finite-dimensional vector space over $\mathbb{K}$. If L is of finite rank, then the dimension of the vector space $(\mathbb{L}_m/L)$ is called the *rank* of L, denoted by $\mathrm{rank}(L)$.

With the notation just introduced, we state our factorization problem.

Problem F. Given a submodule L of $\mathbb{L}_m$ with finite rank, find all the submodules (in $\mathbb{L}_m$) containing L.

Let $\mathbb{F}$ be a differential field containing $\mathbb{K}$. For a vector $\vec{z} = (z_1, \ldots, z_m) \in \mathbb{F}^m$ and $\gamma = \theta y_i \in \Gamma$, $\gamma \vec{z}$ is understood as $\theta(z_i)$. The vector $\vec{z}$ is a solution of f in (1) if $f(\vec{z}) = \sum_{\gamma \in \Gamma} f_\gamma (\gamma \vec{z}) = 0$. For a set S in $\mathbb{L}_m$, $\vec{z}$ is a solution of S if every element of S annihilates $\vec{z}$. A system of PDE's $\{f_1 = 0, \ldots, f_k = 0\}$, where $f_1, \ldots, f_k$ are in $\mathbb{L}_m$, has the same solutions as $(\{f_1, \ldots, f_k\})$. So we study submodules instead of systems of linear homogeneous PDE's. This point of view enables us to make statements concise.

To speak about all solutions of a submodule, we fix a differential extension field $\widetilde{\mathbb{K}}$ of $\mathbb{K}$ containing all solutions of any submodule of $\mathbb{L}_m$ with finite rank. Such a differential field always exists (Kolchin, 1973, p. 133). An element a of $\widetilde{\mathbb{K}}$ is called a ∂_i-constant if $\partial_i(a)$ is equal to zero. For example, all elements of $\mathbb{K}_i$ are ∂_i-constants. If $\partial_i(a) = 0$, for all i with $1 \leq i \leq n$, then a is called a constant. All constants of $\widetilde{\mathbb{K}}$ form a field denoted by $\widetilde{\mathbb{C}}$.

For a submodule L of $\mathbb{L}_m$, the set of solutions of L in $\widetilde{\mathbb{K}}$ is denoted by $\operatorname{sol}(L)$. This set is a vector space over $\widetilde{\mathbb{C}}$. Proposition 2 in Kolchin (1973, p. 151) implies that $\operatorname{rank}(L)$ is equal to the dimension of $\operatorname{sol}(L)$ over $\widetilde{\mathbb{C}}$ if either $\operatorname{rank}(L)$ or $\dim_{\widetilde{\mathbb{C}}} \operatorname{sol}(L)$ is finite (see also Theorem 1.4.22 in Saito et al. (2000)). If a submodule M properly contains a submodule L with finite rank, then $\operatorname{sol}(M)$ is properly contained in $\operatorname{sol}(L)$. A submodule of rank k containing L is called a rank k factor of L. A solution to Problem F is an algorithm for computing all factors whose ranks are lower than $\operatorname{rank}(L)$.

Remark 2.1. It is convenient to introduce the field $\widetilde{\mathbb{K}}$ to describe solutions of submodules, although all the calculations will be performed in $\mathbb{K}$. It is not required to determine the field $\widetilde{\mathbb{K}}$ explicitly in the algorithms in this paper.

We recall the notion of hyperexponential elements which play a key role in our factorization algorithm. The logarithmic derivative w.r.t. x_i of a nonzero element a of $\widetilde{\mathbb{K}}$ is the ratio of $\partial_i(a)$ and a, which is denoted by $(\partial_i \log)a$. The following rules are obvious: For $a, b \in \widetilde{\mathbb{K}}$ and $i, j \in \{1, \ldots, n\}$,

- $(\partial_i \log)(ab) = (\partial_i \log)a + (\partial_i \log)b$,
- $(\partial_i \log)\left(a^{-1}\right) = -(\partial_i \log)a$,
- $\partial_j(\partial_i \log)a = \partial_i(\partial_j \log)a$.

A nonzero element a of $\widetilde{\mathbb{K}}$ is said to be *hyperexponential* w.r.t. x_i over $\mathbb{K}$ if the ith logarithmic derivative $(\partial_i \log)a$ belongs to $\mathbb{K}$. The element a is said to be hyperexponential if $(\partial_i \log)a$ belongs to $\mathbb{K}$ for all i with $1 \leq i \leq n$. Examples of hyperexponential elements are rational, exponential and certain algebraic functions, e.g. functions defined by $z^k = a$ where $k \in \mathbb{N}$ and $a \in \mathbb{K}$. The product and ratio of two hyperexponential elements remain hyperexponential; while their sum is not necessarily hyperexponential.

Since $\mathbb{L}_1 = \mathbb{D}y_1$, $\mathbb{L}_1$ can be identified with $\mathbb{D}$ and a submodule of $\mathbb{L}_1$ can be identified with an ideal of $\mathbb{D}$. We will use the ring $\mathbb{D}$ and ideals instead of the module $\mathbb{L}_1$ and submodules, respectively. In doing so, we can omit the indeterminate y_1 which helps little to describe solutions in the case $m = 1$.

To solve Problem F, we need to solve

Problem H1. Given an ideal of $\mathbb{D}$ with finite rank, find all its hyperexponential solutions.

As a byproduct, we will solve

Problem H. Given a submodule of $\mathbb{L}_m$ with finite rank, find all its nontrivial solutions whose components are either zero or hyperexponential.

In the rest of this section, we briefly review the notion of Janet (Gröbner) bases of a submodule, which is fundamental for us to compute rank of submodules, to reduce Problem H1 to the problem of computing hyperexponential solutions of linear ODEs, and to form factor candidates in our factorization algorithm. The notion of Janet bases in this paper may be viewed either as an extension of Gröbner basis for ideals in $\mathbb{D}$ (Galligo, 1985; Kandru-Rody and Weispfenning, 1990; Chyzak and Salvy, 1998) to submodules in $\mathbb{L}_m \cong \mathbb{D}^m$, or as a specialization of nonlinear characteristic sets (Kolchin, 1973; Wu, 1989) to linear homogeneous differential polynomials.

A total ordering $\prec$ on Γ is admissible if $\gamma \prec \theta\gamma$, and $\gamma_1 \prec \gamma_2 \Longrightarrow \theta\gamma_1 \prec \theta\gamma_2$, for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $\theta \in \Theta\backslash\{1\}$. Fix an admissible total ordering $\prec$ on Γ. For nonzero f in $\mathbb{L}_m$, the highest derivative appearing in f is called the *leading derivative* or *leader* of f and denoted by $\mathrm{lder}(f)$. The coefficient of $\mathrm{lder}(f)$ is denoted by $\mathrm{lc}(f)$. For $f, g \in \mathbb{L}_m$ with $g \neq 0$, f is *reduced* w.r.t. g if any derivative of $\mathrm{lder}(g)$ does not appear in f. Moreover, f is reduced w.r.t. a subset S of nonzero elements of $\mathbb{L}_m$ if f is reduced w.r.t every element of S. A set $S \subset \mathbb{L}_m$ is said to be *autoreduced* if its elements are reduced pairwise. Every autoreduced set is finite (Kolchin, 1973, Chapter 0, Section 17).

Let L be a nontrivial submodule, and Ξ be the set consisting of all leading derivatives of nonzero elements of L. There exists an autoreduced set A in Ξ such that any element of Ξ is a derivative of some element of A. In fact, it is not hard to prove that A is unique. We denote it by $\mathrm{lder}(L)$. A derivative is called a *parametric derivative* of L if it is reduced w.r.t. $\mathrm{lder}(L)$. The set of parametric derivatives of L is denoted by $\mathrm{pder}(L)$.

A finite subset J of L is called a *Janet basis* if every nonzero element of L is not reduced w.r.t. J. We attribute bases of this type to Janet because he appears to be the first person who conceived a thorough reduction–completion process for PDE's (Janet, 1920). There are a number of ways to construct Janet bases from a given finite basis for L. The following seems to be the most concise one, which can be viewed either as an extension of Gröbner bases in $\mathbb{D}$ to $\mathbb{L}_m$ or as a specialization of coherent autoreduced sets (Rosenfeld, 1959; Boulier et al., 1995) to linear differential polynomials.

Let f_1 and f_2 be nonzero elements of $\mathbb{L}_m$ with respective leading derivatives $\theta_1 y$ and $\theta_2 y$, where $y \in \{y_1, \ldots, y_m\}$ and $\theta_1, \theta_2 \in \Theta$. There exist ϕ_1 and ϕ_2 in Θ such that $\phi_1\theta_1 = \phi_2\theta_2$ with minimal orders. The Δ-*polynomial* of f_1 and f_2 is defined to be $\Delta(f_1, f_2) = (\mathrm{lc}(f_2)(\phi_1 f_1) - \mathrm{lc}(f_1)(\phi_2 f_2))$. An autoreduced set A is said to be coherent if, for every pair f_1, f_2 in A as described above, $\Delta(f_1, f_2)$ can be written as a $\mathbb{K}$-linear combination of derivatives of elements of A, in which each derivative has its leader lower than $\phi_1\theta_1 y$. For the submodule L there exists a unique Janet basis (up to some multiplicative scalars of $\mathbb{K}$) which is a coherent autoreduced set. Such a basis is also called the reduced Gröbner basis for L in the literature.

We may read off $\mathrm{lder}(L)$ and $\mathrm{pder}(L)$ from a Janet basis for L. If L is of finite rank, then $\mathrm{rank}(L) = |\mathrm{pder}(L)|$. For an ideal $I \subset \mathbb{D}$ with finite rank, the generator of the ideal $I \cap \mathbb{K}[\partial_i]$ in $\mathbb{K}[\partial_i]$, where $1 \leq i \leq n$, can be computed by Janet basis computation and linear algebra.

3. Hyperexponential solutions of ideals with finite rank

We describe an algorithm to solve Problem H1. Throughout this section, we let I be an ideal of $\mathbb{D}$ with finite rank. The set of hyperexponential solutions of I is denoted by $\mathcal{H}(I)$. For $i = 1, \ldots, n$, let I_i be the ideal $I \cap \mathbb{K}[\partial_i]$ in $\mathbb{K}[\partial_i]$.

To illustrate the idea of our algorithm, let us compute rational solutions of I. First, compute rational solutions of each I_i. Suppose a basis of rational solutions of I_i is $\{r_{i1}, \ldots, r_{im_i}\}$, $i = 1, \ldots, n$. Let q be the common denominator of all r_{ij}, where $1 \le j \le m_i$ and $1 \le i \le n$. Then q is a common denominator of all rational solutions of I. In particular, we can write r_{ij} as p_{ij}/q where p_{ij} are polynomials in x_i, $1 \le j \le m_i$ and $1 \le i \le n$. Then every rational solution of I can be written as p/q, where p is a polynomial with degree in x_i no more than the maximum of the degrees of $p_{i1}, \ldots, p_{im_i}$ in x_i, for all i with $1 \le i \le n$. Hence, we may find p by solving a linear algebraic system over $\overline{\mathbb{Q}}$. Other methods for computing polynomial and rational solutions of an ideal with finite rank can be found in Chyzak (2000) and Oaku et al. (2001).

If $h \in \mathcal{H}(I)$, then it is a hyperexponential solution of I_i w.r.t. x_i. We shall

(1) compute hyperexponential solutions of I_i w.r.t. x_i, $i = 1, \ldots, n$,

(2) combine these solutions to recover $\mathcal{H}(I)$.

The first step is carried out by finding rational solutions (in $\mathbb{K}$) of the Riccati equation associated with the generator of I_i. It is possible to adapt the classical algorithm (Singer, 1991; Bronstein, 1992) to find these solutions (Li and Schwarz, 2001, Section 4). The second step hinges on a structure theorem of $\mathcal{H}(I)$ (Proposition 3.4) and the notion of common hyperexponential associates (Section 3.2). It might be possible to find hyperexponential solutions of one of the I_i's and then design a back-substitution to get $\mathcal{H}(I)$. But our investigation in this direction has been unsuccessful so far, because we need to differentiate integrals w.r.t. parameters, and deal with arbitrary irrational functions.

This section is organized as follows. Section 3.1 proves a structure theorem on $\mathcal{H}(I)$. Section 3.2 introduces the notion of common hyperexponential associates. Section 3.3 presents an algorithm for computing $\mathcal{H}(I)$.

3.1. *Structure of hyperexponential solutions*

For a nonzero element $u \in \widetilde{\mathbb{K}}$, its *logarithmic gradient* $((\partial_1 \log)u, \ldots, (\partial_n \log)u)$ is denoted by $(\nabla \log)u$. From the rules for logarithmic derivatives it follows that $(\nabla \log)(uv) = (\nabla \log)(u) + (\nabla \log)(v)$ and $(\nabla \log)(u^{-1}) = -(\nabla \log)(u)$.

An element u of $\widetilde{\mathbb{K}}$ is hyperexponential over $\mathbb{K}$ if and only if $(\nabla \log)u$ belongs to $\mathbb{K}^n$. A vector $\vec{v} = (v_1, \ldots, v_n)$ in $\mathbb{K}^n$ is said to be *compatible* if $\partial_i v_j = \partial_j v_i$ for all i, j with $1 \le i < j \le n$. If u is hyperexponential over $\mathbb{K}$, then $(\nabla \log)u$ is compatible. Conversely, if $\vec{v}$ is compatible, then any nonzero solution of the ideal $(\partial_1 - v_1, \ldots, \partial_n - v_n)$ is hyperexponential over $\mathbb{K}$. So a hyperexponential element with logarithmic gradient $\vec{v}$ is also denoted by

$$\exp\left(\int v_1 \, \mathrm{d}x_1 + \cdots + v_n \, \mathrm{d}x_n\right). \tag{2}$$

Note that the notation in (2) denotes hyperexponential elements which may differ from a nonzero multiplicative constant in $\widetilde{\mathbb{C}}$.

The following technical lemma will be frequently used in the sequel.

Lemma 3.1. *Let nonzero $r_1, \ldots, r_n$ belong to $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. If*

$$\vec{w} = ((\partial_1 \log) r_1, \ldots, (\partial_n \log) r_n)$$

is a compatible vector of $\mathbb{K}^n$, then $\vec{w}$ is a logarithmic gradient of an element of $\mathbb{K} = \overline{\mathbb{Q}}(x_1, \ldots, x_n)$.

Proof. The proof will rely on the following claim. Let i be in $\{1, \ldots, n\}$. and a in $\mathbb{K}$. If the operator $b = (\partial_i - a)$ has a nonzero solution in $\mathbb{F}(x_i)$, where $\mathbb{F}$ is the field consisting of all ∂_i-constants, then b has a nonzero solution in $\mathbb{K}$.

To show the claim, let nonzero s in $\mathbb{F}(x_i)$ be a solution of b, and the denominator of a be $p \in \mathbb{K}_i[x_i]$. Since every finite pole of s is a root of p, there exists a positive integer k such that p^k is a multiple of the denominator of s. Therefore, s can be written as the ratio of q and p^k, where q belongs to $\mathbb{F}[x_i]$. Let $\deg_{x_i} q = d$ and set $t = (q_d x_i^d + \cdots + q_0)/p^k$, where $q_d, \ldots, q_0$ are unspecified ∂_i-constants. Applying b to t yields a linear system in $q_d, \ldots, q_0$ with coefficients in $\mathbb{K}_i$. This system has a nonzero solution consisting of the coefficients of q, so the system has a nonzero solution in $\mathbb{K}_i$. Hence, b has a nonzero solution in $\mathbb{K}$.

For each i with $1 \le i \le n$, the first-order operator $f_i = (\partial_i - (\partial_i \log) r_i)$ has a solution r_i in $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. Since $f_i \in \mathbb{K}[\partial_i]$, it has a nonzero solution in $\mathbb{K}$ by the claim. Thus, we may further assume that $r_1, \ldots, r_n$ belong to $\mathbb{K}$.

We proceed by induction on n. The lemma clearly holds when $n = 1$. Assume that the lemma holds for $(n-1)$. For each i with $1 \le i \le (n-1)$,

$$\partial_n (\partial_i \log) r_i - \partial_i (\partial_n \log) r_n = 0$$

implies that

$$\partial_n \left(\underbrace{(\partial_i \log) r_i - (\partial_i \log) r_n}_{u_i} \right) = 0,$$

so that u_i belongs to $\mathbb{K}_n$. Since the operator $\partial_i - u_i$ has a solution r_i/r_n in $\mathbb{K}$, it has a solution v_i in $\mathbb{K}_n$ by the claim. Since the compatible vector $((\partial_1 \log) r_1, \ldots, (\partial_{n-1} \log) r_{n-1})$ equals

$$((\partial_1 \log) r_n, \ldots, (\partial_{n-1} \log) r_n) + ((\partial_1 \log) v_1, \ldots, (\partial_{n-1} \log) v_{n-1}),$$

the vector $((\partial_1 \log) v_1, \ldots, (\partial_{n-1} \log) v_{n-1}) \in \mathbb{K}_n^{n-1}$ is compatible. The induction hypothesis then implies that there exists g in $\mathbb{K}_n$ such that

$$((\partial_1 \log) g, \ldots, (\partial_{n-1} \log) g) = ((\partial_1 \log) v_1, \ldots, (\partial_{n-1} \log) v_{n-1}).$$

It follows from a direct verification that $(\nabla \log)(r_n g)$ is equal to $\vec{w}$. $\square$

Two compatible vectors are said to be *equivalent* if their difference is a logarithmic gradient of some element of $\mathbb{K}$. Two hyperexponential elements are said to be *equivalent* if their logarithmic gradients are equivalent.

Lemma 3.2. *Let u and v be hyperexponential over $\mathbb{K}$. Then the following statements are equivalent.*

(1) *u and v are equivalent;*
(2) *u/v is the product of a constant in $\widetilde{\mathbb{C}}$ and a rational function in $\mathbb{K}$.*
(3) *u and v are linearly dependent over $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$.*

Proof. If the first assertion holds, there is $r \in \mathbb{K}$ such that $(\nabla\log)(r)$ is equal to $((\nabla\log)u - (\nabla\log)v)$, and so the fraction u/v is a solution of the ideal $(\partial_1 - (\partial_1\log)r, \ldots, \partial_n - (\partial_n\log)r)$. It follows that $u/v = cr$ for some $c \in \widetilde{\mathbb{C}}$. The second assertion clearly implies the last. To show that the last implies the first, let $u = rv$ for some $r \in \widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. Then $(\nabla\log)(r)$ is equal to $((\nabla\log)(u) - (\nabla\log)(v)) \in \mathbb{K}^n$. It follows from Lemma 3.1 that there is q in $\mathbb{K}$ with $(\nabla\log)(q) = ((\nabla\log)(u) - (\nabla\log)(v))$, and the lemma follows. □

Next, we characterize mutually inequivalent hyperexponential elements.

Proposition 3.3. *Let $h_1, \ldots, h_m$ be hyperexponential elements over $\mathbb{K}$. The elements $h_1, \ldots, h_m$ are mutually inequivalent if and only if $h_1, \ldots, h_m$ are linearly independent over $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$.*

Proof. The necessity follows from Lemma 3.2. We prove the sufficiency by induction on m. If $m = 2$, then the proposition holds by Lemma 3.2. Assume that the result is proved for lower values of m. Suppose that $h_1, \ldots, h_m$ are mutually inequivalent but linearly dependent over $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. By a possible rearrangement of indexes, we have

$$h_m = q_1 h_1 + q_2 h_2 + \cdots + q_{m-1} h_{m-1} \tag{3}$$

for some $q_1, q_2, \ldots, q_{m-1} \in \widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. Since $h_1, h_2, \ldots, h_{m-1}$ are linearly independent over $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$ by the induction hypothesis, we deduce that $q_1 h_1, \ldots, q_{m-1} h_{m-1}$ are linearly independent over $\widetilde{\mathbb{C}}$. Then Theorem 1 in Kolchin (1973, p. 86) implies that there exist derivatives $\theta_1, \theta_2, \ldots, \theta_{m-1}$ in Θ such that $W = \det(\theta_i(q_j h_j))$ is nonzero, where $1 \le i \le m-1$ and $1 \le j \le m-1$. Since the h_i's are hyperexponential, there exist r_{ij} in $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$ such that $\theta_i(q_j h_j) = r_{ij} h_j$, for each i and each j. Applying $\theta_1, \theta_2, \ldots, \theta_{m-1}$ to (3) then yields a linear system

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,m-1} \\ r_{21} & r_{22} & \cdots & r_{2,m-1} \\ \cdot & \cdot & \cdots & \cdot \\ r_{m-1,1} & r_{m-1,2} & \cdots & r_{m-1,m-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m-1} \end{pmatrix} = \begin{pmatrix} r_{1m} h_m \\ r_{2m} h_m \\ \vdots \\ r_{m,m-1} h_m \end{pmatrix}$$

whose coefficient matrix (r_{ji}) is of full rank, because $(\prod_{i=1}^{m} h_i)\det(r_{ij}) = W$ is nonzero. Solving this system, we get $h_i = s_i h_m$, where $s_i \in \widetilde{\mathbb{C}}(x_1, \ldots, x_n)$. So h_i and h_m are equivalent by Lemma 3.2, a contradiction. □

To decide if a finite number of hyperexponential elements over $\mathbb{K}$ are linearly dependent over $\widetilde{\mathbb{C}}(x_1, \ldots, x_n)$, we need only to decide if there exist two equivalent elements among the given hyperexponential elements by Proposition 3.3. To decide if two hyperexponential elements f and g are equivalent, we need to check if the logarithmic gradient of f/g is a logarithmic gradient of some elements of $\mathbb{K}$. By Lemma 3.1 it suffices to decide if the rational function $r_i = (\partial_i\log)(f/g)$ is equal to $(\partial_i \log q_i)$ for some $q_i \in \mathbb{K}$, for $i = 1, \ldots, n$. It is straightforward to show that such q_i exists if and only if the squarefree partial fraction decomposition of r_i w.r.t. x_i is in the form $\sum_j k_{ij}\left(\partial_i(p_{ij})/p_{ij}\right)$ where the k_{ij}'s are nonzero integers and the p_{ij}'s are polynomials in $\mathbb{K}$.

Let $\mathbb{P}$ stand for the ring $\overline{\mathbb{Q}}[x_1,\ldots,x_n]$. A finite set of polynomials in $\mathbb{P}$ is said to be *independent* over a subfield $\mathbb{F} \subset \mathbb{K}$ if its elements are linearly independent over $\mathbb{F}$. Let P be an independent set over $\overline{\mathbb{Q}}$ and g a hyperexponential element over $\mathbb{K}$. We denote by $\mathcal{H}_{(g,P)}$ the set consisting of the nonzero elements in the form $cg(\sum_{p\in P} c_p p)$, where c is in $\widetilde{\mathbb{C}}$ and the c_p in $\overline{\mathbb{Q}}$. Clearly, all elements of $\mathcal{H}_{(g,P)}$ are hyperexponential over $\mathbb{K}$ and equivalent to g. The following proposition describes the structure of $\mathcal{H}(I)$.

Proposition 3.4. *If I is an ideal of $\mathbb{D}$ with finite rank d, then there is a finite number of mutually inequivalent hyperexponential elements $g_1,\ldots,g_k$ and independent sets $P_1,\ldots,P_k$ in $\mathbb{P}$ over $\overline{\mathbb{Q}}$ s.t. $\mathcal{H}(I) = \mathcal{H}_{(g_1,P_1)} \cup\cdots\cup \mathcal{H}_{(g_k,P_k)}$, in which the union is disjoint. The sum of $|P_1|,\ldots,|P_k|$ is not more than d.*

Proof. The equivalence relation gives rise to a partition of $\mathcal{H}(I)$. By Proposition 3.3 there are only finitely many equivalence classes in $\mathcal{H}(I)$. Hence, we have the partition $\mathcal{H}(I) = \mathcal{H}_1\cup\cdots\cup\mathcal{H}_k$, in which $\mathcal{H}_i$ stands for an equivalence class. Let h_i belong to $\mathcal{H}_i$ for $i = 1,\ldots,k$. For every $f \in \mathcal{H}_i$ there exists $c_f \in \widetilde{\mathbb{C}}$ and $r_f \in \mathbb{K}$ such that $f = c_f r_f h_i$ by the second assertion of Lemma 3.2. Pick up a maximal $\overline{\mathbb{Q}}$-linearly independent set Q_i from all such r_f. The elements of Q_i are also $\widetilde{\mathbb{C}}$-linearly independent (Kolchin, 1973, pp. 86, 87). The set Q_i is finite, for otherwise $\mathcal{H}_i$ would contain infinitely many $\widetilde{\mathbb{C}}$-linearly independent elements. Let $Q_i = \{\frac{p_1}{q},\ldots,\frac{p_{d_i}}{q}\}$, where $p_1,\ldots,p_{d_i}, q \in \mathbb{P}$, and $g_i = h_i/q$. Then g_i is hyperexponential and equivalent to h_i. Let $P_i = \{p_1,\ldots,p_{d_i}\}$. Then $\mathcal{H}_{(g_i,\,P_i)} = \mathcal{H}_i$. The first assertion then follows from the partition of $\mathcal{H}(I)$. The sum of $|P_1|,\ldots,|P_k|$ is no more than d by Proposition 3.3. □

By computing the hyperexponential solutions of the ideal I, we mean to compute mutually inequivalent hyperexponential elements $g_1,\ldots,g_m$, and independent sets $P_1,\ldots,P_k$ such that $\mathcal{H}(I) = \mathcal{H}_{(g_1,P_1)} \cup\cdots\cup \mathcal{H}_{(g_k,P_k)}$.

3.2. Common hyperexponential associates

Two hyperexponential elements of $\widetilde{\mathbb{K}}$ w.r.t. x_i are said to be *equivalent w.r.t. x_i* if the difference of their logarithmic derivatives w.r.t. x_i is a logarithmic derivative of some element in $\mathbb{K}$ w.r.t. x_i. For $i = 1,\ldots,n$, we let h_i be hyperexponential w.r.t. x_i in this subsection. A hyperexponential element h over $\mathbb{K}$ is called a *common hyperexponential associate* of $h_1,\ldots,h_n$ if h is equivalent to h_i w.r.t. x_i, for $i = 1,\ldots,n$. In other words, a common hyperexponential associate h of $h_1,\ldots,h_n$ is a hyperexponential element of $\mathbb{K}$ such that

$$h_1 = c_1 r_1 h, \ldots, h_n = c_n r_n h \tag{4}$$

where c_i is a ∂_i-constant and r_i belongs to $\mathbb{K}$, $i = 1,\ldots,n$. For simplicity, we shall use the term *common associates* instead of *common hyperexponential associates* if no confusion arises.

Lemma 3.5. *If $h_1, \ldots, h_n$ have a common associate, then there exists a common associate f of $h_1, \ldots, h_n$ such that*

$$h_1 = c_1 p_1 f, \ldots, h_n = c_n p_n f, \tag{5}$$

where c_i is a ∂_i-constant and p_i belongs to $\mathbb{P}$, $i = 1, \ldots, n$. In addition, any two common associates of $h_1, \ldots, h_n$ are equivalent.

Proof. Let h be a common associate of $h_1, \ldots, h_n$ satisfying (4). Write $r_1 = (p_1/q), \ldots, r_n = (p_n/q)$, where $p_1, \ldots, p_n$ and q belong to $\mathbb{P}$. Then f in (5) can be chosen as h/q. Let f and g be two common associates of $h_1, \ldots, h_n$. By (4), $f = b_i s_i g$, where b_i is a ∂_i-constant, s_i belongs to $\mathbb{K}$, and $i = 1, \ldots, n$. It follows that $(\nabla \log) f - (\nabla \log) g = ((\partial_1 \log) s_1, \ldots, (\partial_n \log) s_n)$, which is compatible. Lemma 3.1 then implies that f and g are equivalent. $\square$

Applying $(\partial_i \log)$ to the equalities in (5), we get

$$\begin{cases} (\partial_1 \log) h_1 = (\partial_1 \log) p_1 + (\partial_1 \log) f \\ \qquad \cdots \\ (\partial_n \log) h_n = (\partial_n \log) p_n + (\partial_n \log) f. \end{cases} \tag{6}$$

Applying ∂_i to the jth equation and ∂_j to the ith equation in (6) and using the equality $\partial_i (\partial_j \log) f = \partial_j (\partial_i \log) f$, we deduce

$$\partial_j (\partial_i \log) p_j - \partial_i (\partial_j \log) p_i = \partial_j (\partial_i \log) h_j - \partial_i (\partial_j \log) h_i, \qquad (1 \le i \le j \le n).$$

Therefore, $(p_1, \ldots, p_n)$ is a polynomial solution of the system

$$\partial_j (\partial_i \log) \left(\frac{z_j}{z_i} \right) = \partial_j (\partial_i \log) \left(\frac{h_j}{h_i} \right), \qquad (1 \le i \le j \le n). \tag{7}$$

Conversely, if (7) has a rational or polynomial solution $(p_1, \ldots, p_n)$, then

$$\left(\underbrace{(\partial_1 \log) h_1 - (\partial_1 \log) p_1}_{f_1}, \ldots, \underbrace{(\partial_n \log) h_n - (\partial_n \log) p_n}_{f_n} \right)$$

is a compatible vector in $\mathbb{K}^n$. A direct calculation shows that

$$f = \exp \left(\int f_1 \, \mathrm{d}x_1 + \cdots + f_n \, \mathrm{d}x_n \right) \tag{8}$$

is a common associate such that (5) holds. Thus, we have proved

Proposition 3.6. *The elements $h_1, \ldots, h_n$ have a common associate if and only if (7) has a nonzero polynomial solution. If (7) has a nonzero polynomial solution $(p_1, \ldots, p_n)$, then f given in (8) is a common associate of $h_1, \ldots, h_n$ such that (5) holds.*

The next corollary indicates a special property of the system (7).

Corollary 3.7. *If (7) has two solutions $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ in $\mathbb{K}^n$, then there exists $r \in \mathbb{K}$ s.t. $(\partial_i \log) p_i - (\partial_i \log) q_i = (\partial_i \log) r$, $i = 1, \ldots, n$.*

Proof. Let $g_i = (\partial_i \log) h_i$, for all i with $1 \le i \le n$. Proposition 3.6 implies that both $H_1 = \exp(\int (g_1 - (\partial_1 \log) p_1)\, dx_1 + \cdots + (g_n - (\partial_n \log) p_n)\, dx_n)$ and $H_2 = \exp(\int (g_1 - (\partial_1 \log) q_1)\, dx_1 + \cdots + (g_n - (\partial_n \log) q_n)\, dx_n)$ are common associates of $h_1, \ldots, h_n$. It follows from Lemmas 3.2 and 3.5 that there exist $c \in \widetilde{\mathbb{C}}$ and $r \in \mathbb{K}$ such that $H_1 = cr H_2$. Applying $(\partial_i \log)$ to this equality yields the corollary. □

To compute polynomial solutions of (7), we need

Algorithm RationalAntiderivative (Find Rational Antiderivative). Given $a_1, \ldots, a_k$ in $\mathbb{K}$, where $1 \le k \le n$, the algorithm finds a nonzero r in $\mathbb{K}$ such that $\{\partial_1 r = a_1, \ldots, \partial_k r = a_k\}$ or determines that no such solutions exist.

1. [Recursive base]. If $k = 1$, apply the Hermite reduction to a_1 w.r.t. x_1 to get $q_1, r_1 \in \mathbb{K}$ such that $a_1 = \partial_1 q_1 + r_1$, where r_1 has a squarefree denominator in x_1 (Geddes et al., 1992; Bronstein, 1997). If r_1 is nonzero, then exit [no such solution exists].
2. [Recursion]. If RationalAntiderivative $(a_1, \ldots, a_{k-1})$ determines that no such solution exists in $\mathbb{K}$, then exit [no such solution exists]. Otherwise, let a solution be r_{k-1} and set $b_k \leftarrow (a_k - \partial_k r_{k-1})$.
3. [Hermite reduction]. If b_k is not in $\overline{\mathbb{Q}}(x_k, x_{k+1}, \ldots, x_n)$, then exit [no such solution exists]. Otherwise, apply the Hermite reduction to b_k w.r.t. x_k to get q_k, h_k in $\mathbb{K}$ such that $b_k = \partial_k q_k + h_k$, where h_k has a squarefree denominator in x_k. If h_k is nonzero, then exit [no such solution exists]. Otherwise, set $r \leftarrow r_{k-1} + q_k$. □

In step 1 the Hermite reduction w.r.t. x_1 enables us to compute a solution of $\partial_1 r = f_1$ in $\mathbb{K}$. Assume that we can compute a nonzero solution r_{k-1} of the system $\{\partial_1 r = a_1, \ldots, \partial_{k-1} r = a_{k-1}\}$ in $\mathbb{K}$. Its rational solutions are in the form

$$r = r_{k-1} + q_k, \qquad \text{where } q_k \in \overline{\mathbb{Q}}(x_k, x_{k+1}, \ldots, x_n). \tag{9}$$

If $\partial_k r_{k-1} = a_k$ then r_{k-1} is what we seek. Otherwise $b_k = (a_k - \partial_k r_{k-1})$ is nonzero. Substituting (9) into $\partial_k r = a_k$, we get $b_k = \partial_k q_k$. Hence, b_k belongs to $\overline{\mathbb{Q}}(x_k, x_{k+1}, \ldots, x_n)$ and q_k can be computed by the Hermite reduction w.r.t. x_k. The algorithm RationalAntiderivative is correct.

To describe the next algorithm, we denote by $\mathbb{P}^*$ the set $\mathbb{P} \backslash \{0\}$ and by E_k the set consisting of the equations in (7) with $1 \le i \le j \le k$, where $1 \le k \le n$.

Algorithm PolynomialRatio (Find a Solution $(p_1, \ldots, p_k)$ of E_k in $(\mathbb{P}^*)^k$). Given $h_1, \ldots, h_k$, where $1 \le k \le n$ and h_i is hyperexponential w.r.t. x_i over $\mathbb{K}$, $i = 1, \ldots, k$, the algorithm finds $(p_1, \ldots, p_k)$, where p_i belongs to $\mathbb{P}^*$, such that E_k holds or determines that E_k has no polynomial solution.

1. [Recursive base]. If $k = 1$, then return 1.
2. [Recursion]. If PolynomialRatio $(h_1, \ldots, h_{k-1})$ finds no polynomial solution, then exit [E_k has no polynomial solution]. Otherwise, let its output be $(p_1, \ldots, p_{k-1})$.
3. [Find a rational solution]. Apply RationalAntiderivative to the system

$$\begin{cases} \partial_1 z = \partial_1 (\partial_k \log) h_k - \partial_k (\partial_1 \log) h_1 + \partial_k (\partial_1 \log) p_1, \\ \cdots \\ \partial_{k-1} z = \partial_{k-1} (\partial_k \log) h_k - \partial_k (\partial_{k-1} \log) h_{k-1} + \partial_k (\partial_{k-1} \log) p_{k-1}. \end{cases} \tag{10}$$

If no rational solution is found, then exit [E_k has no polynomial solution]. Otherwise, let its output be z.

4. [Partial fraction decomposition]. Write z as

$$(\partial_k \log) q - (\partial_k \log) p + g \tag{11}$$

where $p, q \in \mathbb{P}$ and $g \in \overline{\mathbb{Q}}(x_k, x_{k+1}, \ldots, x_n)$ by squarefree partial fraction decomposition w.r.t. x_k. If (11) cannot hold, then exit [E_k has no polynomial solution]. Otherwise, return $(p_1 p, \ldots, p_{k-1} p, q)$. $\square$

To see the correctness of this algorithm, we need to show

- If PolynomialRatio outputs $(p_1, \ldots, p_k) \in (\mathbb{P}^*)^k$, $(p_1, \ldots, p_k)$ solves E_k.
- If E_k has solutions in $(\mathbb{P}^*)^n$, PolynomialRatio produces such a solution.

The algorithm is clearly correct when $k = 1$. We proceed by induction on k. Assume that PolynomialRatio outputs a vector $\vec{u} = (p_1 p, \ldots, p_{k-1} p, q)$, where $p_1, \ldots, p_{k-1}$ are produced by step 3, and p, q by step 4. $(p_1 p, \ldots, p_{k-1} p)$ solves E_{k-1} by the multiplicative rule of logarithmic differentiation and induction hypothesis. It remains to verify that $(p_1 p, \ldots, p_{k-1} p, q)$ solves

$$\partial_k (\partial_i \log) z_k - \partial_i (\partial_k \log) z_i = \partial_k (\partial_i \log) h_k - \partial_i (\partial_k \log) h_i \qquad (1 \le i \le k-1).$$

For all i with $1 \le i \le k-1$, we calculate

$$\begin{aligned} \partial_k (\partial_i \log) q - \partial_i (\partial_k \log)(p p_i) &= \partial_i ((\partial_k \log) q - (\partial_k \log) p) - \partial_i (\partial_k \log) p_i \\ &\overset{(11)}{=} \partial_i z - \partial_k (\partial_i \log) p_i \overset{(10)}{=} \partial_k (\partial_i \log) h_k - \partial_i (\partial_k \log) h_i . \end{aligned}$$

To show the second assertion, let $(s_1, \ldots, s_k)$ in $(\mathbb{P}^*)^k$ be a solution of E_k. We proceed again by induction on k. The second assertion clearly holds for $k = 1$. Assume that it holds for $(k-1)$. Then we get a polynomial solution $(p_1, \ldots, p_{k-1})$ in step 2. By Corollary 3.7 there exists r in $\mathbb{K}$ s.t.

$$(\partial_i \log) p_i - (\partial_i \log) s_i = (\partial_i \log) r \qquad i = 1, \ldots, k-1. \tag{12}$$

For $i = 1, \ldots, k-1$, we compute

$$\begin{aligned} 0 &\overset{(7)}{=} (\partial_k (\partial_i \log) s_k - \partial_i (\partial_k \log) s_i) - (\partial_k (\partial_i \log) h_k - \partial_i (\partial_k \log) h_i) \\ &= (\partial_k (\partial_i \log)(r s_k) - \partial_i (\partial_k \log)(r s_i)) - (\partial_k (\partial_i \log) h_k - \partial_i (\partial_k \log) h_i) \\ &\overset{(12)}{=} (\partial_k (\partial_i \log)(r s_k) - \partial_i (\partial_k \log) p_i) - (\partial_k (\partial_i \log) h_k - \partial_i (\partial_k \log) h_i). \end{aligned}$$

It follows that $(\partial_k \log)(r s_k)$ is a rational solution of (10). Hence, RationalAntiderivative returns a rational function z in the third step. Since

$$z = \left(\frac{\partial_k s_k}{s_k} + \frac{\partial_k r}{r} \right) + g \qquad \text{where } g \in \overline{\mathbb{Q}}(x_k, x_{k+1}, \ldots, x_n),$$

we get (11). The second assertion holds.

Example 3.1. Find a common associate of $h_i = \exp(\int u_i \, dx_i)$, where $1 \le i \le 3$,

$$u_1 = \frac{x_1 x_2^2 x_3 + 2x_1^2 x_2 x_3 - x_2 - x_1}{x_1^2 (x_2 + x_1) x_2 x_3},$$
$$u_2 = \frac{-x_1 x_2 x_3^2 + 2x_1 x_2^2 x_3 + x_3 - x_2}{(x_2 - x_3) x_2^2 x_1 x_3},$$

and $u_3 = (3x_1x_2x_3 - 1)/x_1x_2x_3^2$. First, PolynomialRatio applies to h_1 and yields 1 as a polynomial solution of u_1 (see step 1 in the algorithm). In step 3, the system (10) becomes $\partial_1 z = \partial_1 u_2 - \partial_2 u_1$ with a rational solution $(x_2+x_1-1)/(x_2+x_1)$, which can be decomposed into

$$(\partial_2 \log)1 - (\partial_2 \log)(x_2 + x_1) + 1.$$

Hence, a polynomial solution of E_2 is $(x_2 + x_1, 1)$. Now the system (10) becomes

$$\{\partial_1 z = \partial_1 u_3 - \partial_3(u_1 - (\partial_1 \log)(x_2 + x_1)),\ \partial_2 z = \partial_2 u_3 - \partial_3 u_2\}$$

with a rational solution $(1 - 2x_3 + 2x_2)/(x_2 - x_3)$, which can be decomposed into

$$(\partial_3 \log)1 - (\partial_3 \log)(x_2 - x_3) + 2.$$

At last, we obtain a polynomial solution $((x_2 + x_1)(x_2 - x_3), x_2 - x_3, 1)$ of E_3. By Proposition 3.6 a common associate of h_1, h_2 and h_3 is

$$\exp\left(\int \left(u_1 - \frac{x_2 - x_3}{x_2 + x_1}\right) dx_1 + \left(u_2 - \frac{1}{x_2 - x_3}\right) dx_2 + u_3 \, dx_3\right).$$

Find a common associate of h_1, h_2 and $h_4 = \exp(\int u_3 + x_1 \, dx_3)$. The algorithm runs exactly the same as before until step 3 for $k = 2$, in which system (10) becomes

$$\begin{cases} \partial_1 z = \partial_1(u_3 + x_1) - \partial_3(u_1 - (\partial_1 \log)(x_2 + x_1)), \\ \partial_2 z = \partial_2(u_3 + x_1) - \partial_3 u_2 \end{cases}$$

with a rational solution $(1 - x_1x_3 + x_1x_2 - x_3 + x_2)/(x_2 - x_3)$, which cannot be written as a logarithmic derivative of a rational function plus a rational function in x_3 alone. Hence, h_1, h_2 and h_4 have no common associate.

3.3. *An algorithm for solving Problem H1*

Let I_i be the ideal $(I \cap \mathbb{K}[\partial_i])$ in $\mathbb{K}[\partial_i]$, for $i = 1, \ldots, n$. Note that I_i is nontrivial because I is of finite rank. We denote by $\mathcal{H}^{(i)}(I_i)$ the set of all hyperexponential solutions of I_i w.r.t. x_i. Clearly,

$$\mathcal{H}(I) \subset \bigcap_{i=1}^{n} \mathcal{H}^{(i)}(I_i). \tag{13}$$

Recall that an independent set over $\mathbb{K}_i$ is a set consisting of finitely many polynomials in $\mathbb{P}$, which are linearly independent over $\mathbb{K}_i$. Viewing $\mathbb{K}$ as an ordinary differential field with derivative operator ∂_i and constant field $\mathbb{K}_i$, we deduce from Proposition 3.4.

Lemma 3.8. *For each i with $1 \le i \le n$, there is a finite number of mutually inequivalent hyperexponential elements $f_{i,1}, \ldots, f_{i,k_i}$ w.r.t. x_i over $\mathbb{K}$, and a finite number of independent sets $P_{i,1}, \ldots, P_{i,k_i}$ over $\mathbb{K}_i$ such that*

$$\mathcal{H}^{(i)}(I_i) = \mathcal{H}^{(i)}_{(f_{i,1}, P_{i,1})} \bigcup \cdots \bigcup \mathcal{H}^{(i)}_{(f_{i,k_i}, P_{i,k_i})}, \tag{14}$$

where $\mathcal{H}^{(i)}_{(f_{i,j}\, P_{i,j})} = \{c f_{i,j}(\sum_{p \in P_{i,j}} c_p p) | c_p \in \mathbb{K}_i, c \text{ is a } \partial_i\text{-constant}\}$. Moreover, the union is disjoint.

For convenience, let F_i be the set consisting of $f_{i1}, \ldots, f_{ik_i}$ in (14). If f belongs to $\mathcal{H}(I)$, then (13) and (14) imply that there exist unique $f_{1j_1} \in F_1, \ldots, f_{nj_n} \in F_n$ such that

$$f \in \bigcap_{i=1}^{n} \mathcal{H}^{(i)}_{(f_{i,j_i}, P_{i,j_i})}. \tag{15}$$

Thus, f is a common associate of $f_{1j_1}, \ldots, f_{nj_n}$. We apply the algorithm PolynomialRatio to each element of $(F_1 \times \cdots \times F_n)$ to get all possible inequivalent common associates of these elements, say $f_1, \ldots, f_k$. Every $f \in \mathcal{H}(I)$ must be equivalent to one and only one of the f_i's. The next lemma tells us how to compute hyperexponential solutions of I that are equivalent to one of the f_i's.

Lemma 3.9. *Let $f_{1k_1}, \ldots, f_{nk_n}$ belong to $F_1, \ldots, F_n$, respectively. Assume that $(p_{1k_1}, \ldots, p_{nk_n})$ is a polynomial solution of (7), in which h_i is replaced by f_{ik_i}, $i = 1, \ldots, n$. Let*

$$h = \exp\left(\int (\partial_1 \log)\left(\frac{f_{1k_1}}{p_{1k_1}}\right) \mathrm{d}x_1 + \cdots + (\partial_n \log)\left(\frac{f_{1k_n}}{p_{1k_n}}\right) \mathrm{d}x_n\right) \tag{16}$$

and e_{i,k_i} be $\max_{p \in P_{i,k_i}}(\deg_{x_i} p)$, where P_{ik_i} is specified in (14), for $i = 1, \ldots, n$. If $f \in \mathcal{H}(I)$ is equivalent to h, then there exists $p \in \mathbb{P}$ such that $f = cph$, for some nonzero $c \in \widetilde{\mathbb{C}}$ and

$$\deg_{x_i} p \le (e_{i,k_i} + \deg_{x_i} p_{i,k_i}), \qquad i = 1, \ldots, n.$$

Proof. The element h is well-defined and is a common associate of $f_{1k_1}, \ldots, f_{nk_n}$ by Proposition 3.6. Let $f \in \mathcal{H}(I)$ be equivalent to h. Then $f \in \mathcal{H}^{(i)}(I_i)$ is equivalent to f_{ik_i} w.r.t. x_i. Thus $(\partial_i \log) f = (\partial_i \log) f_{i,k_i} + (\partial_i \log) q_{i,k_i}$, where q_{i,k_i} is a $\mathbb{K}_i$-linear combination of elements of P_{i,k_i} by Lemma 3.8. It follows that

$$\begin{aligned}(\partial_i \log) f &= (\partial_i \log)\frac{f_{i,k_i}}{p_{i,k_i}} + (\partial_i \log)(q_{i,k_i} p_{i,k_i}) \\ &= (\partial_i \log) h + (\partial_i \log)(q_{i,k_i} p_{i,k_i}).\end{aligned}$$

So, the vector

$$((\partial_1 \log)(q_{1,k_1} p_{1,k_1}), \ldots, (\partial_n \log)(q_{n,k_n} p_{n,k_n})) = (\nabla \log) f - (\nabla \log) h \tag{17}$$

is compatible. By Lemma 3.1 there exists $p \in \mathbb{K}$ such that

$$(\nabla \log) p = ((\partial_1 \log)(q_{1,k_1} p_{1,k_1}), \ldots, (\partial_n \log)(q_{n,k_n} p_{n,k_n})). \tag{18}$$

Thus, $p = c_i(q_{1,k_1} p_{1,k_1})$ for some ∂_i-constant c_i. Consequently, p is a polynomial in x_i with degree less than or equal to $(e_{i,k_i} + \deg_{x_i} p_{i,k_i})$, for $i = 1, \ldots, n$, and thus p belongs to $\mathbb{P}$. The equalities (17) and (18) imply $(\nabla\log)(f/ph) = 0$. $\square$

Example 3.2. Consider the ideal $I \subset \mathbb{K}[\partial_1, \partial_2]$ with rank four, generated by

$$f_1 = \partial_1^3 + \frac{x_2^2 + 6x_1^2 - 6x_1x_2}{2x_1^3 - x_2x_1^2}\partial_1^2,$$

$$f_2 = \partial_2^3 + \frac{3x_1 - 2x_2}{x_1^2 - x_1x_2}\partial_2^2 + \frac{2x_1 - x_2}{x_1^3 - x_1^2x_2}\partial_2.$$

A Janet basis computation reveals that $I_1 = (f_1)$ in $\mathbb{K}[\partial_1]$ and $I_2 = (f_2)$ in $\mathbb{K}[\partial_2]$. Using the algorithms mentioned at the beginning of Section 3 or the `expsols` function in the Maple package `DEtools`, we find that the hyperexponential solutions of I_1 are

$$(u_1 + u_2x_1)\underbrace{\exp\left(\int 0\,\mathrm{d}x_1\right)}_{\alpha_1}, \qquad u_3\underbrace{\exp\left(\int \frac{x_2}{x_1^2}\,\mathrm{d}x_1\right)}_{\alpha_2},$$

where u_1, u_2 and u_3 are ∂_1-constants. The hyperexponential solutions of I_2 are

$$\underbrace{v_1\exp\left(\int 0\,\mathrm{d}x_2\right)}_{\beta_1}, \qquad (v_2 + v_3x_2^2)\underbrace{\exp\left(\int \frac{-1}{x_1}\,\mathrm{d}x_2\right)}_{\beta_2},$$

where v_1, v_2 and v_3 are ∂_2-constants. Applying the algorithm PolynomialRatio to (α_i, β_j) with $i, j = 1, 2$, we see that α_1, β_1 have a common associate $f_1 = 1$, and α_2, β_2 have a common associate

$$f_2 = \exp\left(\int \frac{x_2}{x_1^2}\,\mathrm{d}x_1 - \frac{1}{x_1}\,\mathrm{d}x_2\right) = \exp\left(-\frac{x_2}{x_1}\right);$$

while neither α_1, β_2 nor $\alpha_2\beta_1$ have any common associate. Note that the algorithm PolynomialRatio (α_i, β_i) outputs $(1, 1)$, for $i = 1, 2$. Lemma 3.9 implies that the ideal I can only have hyperexponential solutions in forms:

$$(c_1 + c_2x_1)\,f_1 \qquad \text{or} \qquad (c_3 + c_4x_2 + c_5x_2^2)\,f_2,$$

where $c_1, \ldots, c_5$ belong to $\overline{\mathbb{Q}}$. These constants can be determined by substituting the respective ansatz into f_1 and f_2. As a matter of fact, the solutions are $(c_1 + c_2x_1)$ and $(c_3 + c_5x_2^2)\,f_2$, where $c_1c_2 \neq 0$ and $c_3c_5 \neq 0$. $\square$

Now, we outline an algorithm for solving Problem H1.

Algorithm HyperexponentialSolutions (Compute Hyperexponential Solutions of an Ideal with Finite Rank). Given an ideal $I = (g_1, \ldots, g_s)$ with finite rank in $\mathbb{D}$, the algorithm computes all hyperexponential solutions of I.

1. [Janet basis]. Compute a Janet basis J for I w.r.t. any term-order.

2. [Linear algebra]. Use J to compute a generator of $I_i = I \cap \mathbb{K}[\partial_i]$, for $i = 1, \ldots, n$.
3. [Solve ODEs]. Find $\mathcal{H}^{(i)}(I_i)$, for $i = 1, \ldots, n$. If one of the $\mathcal{H}^{(i)}(I_i)$ is empty, then exit [no such solution exists]. Otherwise, write $\mathcal{H}^{(i)}(I_i)$ as (14) and set $F_i \leftarrow \{f_{i,1}, \ldots, f_{i,k_i}\}$, for $i = 1, \ldots, n$.
4. [Common associate]. Apply the algorithm PolynomialRatio to each vector in $(F_1 \times \cdots \times F_n)$ to construct common associates. If no common associate can be constructed, exit [no such solution exists]. Otherwise, set the constructed common associates to be $f_1, \ldots, f_k$, respectively.
5. [Solution candidates]. Apply Lemma 3.9 to each of the f_j to construct polynomial p_j with unspecified constants such that any element of $\mathcal{H}(I)$ can be expressed as $p_j f_j$ for some j with $1 \le j \le k$.
6. [True solutions]. For each j with $1 \le j \le k$, the system

$$\{g_1(p_j f_j) = 0, \ldots, g_s(p_j f_j) = 0\}$$

gives rise to a *linear* homogeneous algebraic system A_j in the coefficients of p_j. Solve A_j to determine all the elements of $\mathcal{H}(I)$ equivalent to f_j. $\square$

Example 3.3. Compute $\mathcal{H}(I)$, where the rank two ideal I is generated by

$$\partial_1^2 - \frac{x_1}{x_1 - 1}\partial_1 + \frac{1}{x_1 - 1}, \qquad \partial_3 - \frac{2x_1x_3 + \frac{1}{2}x_1}{x_1x_3 - x_3}\partial_1 + \frac{2x_3 + \frac{1}{2}x_1}{x_1x_3 - x_3},$$
$$\partial_2 + \frac{x_1}{x_2(x_1x_2 - x_2)}\partial_1 - \frac{x_1}{x_2(x_1x_2 - x_2)}.$$

Step 2 in HyperexponentialSolutions yields

$$I_1 = \left(\partial_1^2 - \frac{x_1}{x_1 - 1}\partial_1 + \frac{1}{x_1 - 1}\right),$$
$$I_2 = \left(\partial_2^2 + \frac{2x_2 - 1}{x_2^2}\partial_2\right),$$
$$I_3 = \left(\partial_3^2 + \frac{3 - 16x_3^2}{8x_3^2 + 2x_3}\partial_3 - \frac{8x_3 + 6}{8x_3^2 + 2x_3}\right).$$

Step 3 gives:

$$\mathcal{H}^{(1)}(I_1) = \mathcal{H}^{(1)}_{(x_1,\{1\})} \cup \mathcal{H}^{(1)}_{(\exp(x_1),\{1\})}, \qquad \mathcal{H}^{(2)}(I_2) = \mathcal{H}^{(2)}_{\left(\exp\left(\frac{-1}{x_2}\right),\{1\}\right)} \cup \mathcal{H}^{(2)}_{(1,\{1\})},$$

and

$$\mathcal{H}^{(3)}(I_3) = \mathcal{H}^{(3)}_{\left(\frac{1}{\sqrt{x_3}},\{1\}\right)} \cup \mathcal{H}^{(3)}_{(\exp(2x_3),\{1\})}.$$

Step 4 gives us eight common associates. Step 5 sets up eight solution candidates:

$$\exp\left(\int\left(\frac{1}{x_1}\,dx_1 - \frac{1}{2x_3}\,dx_3\right)\right), \qquad \exp\left(\int 1\,dx_1 + \frac{1}{x_2^2}\,dx_2 + 2\,dx_3\right),$$

$$\exp\left(\int\left(1\,\mathrm{d}x_1-\frac{1}{2x_3}\,\mathrm{d}x_3\right)\right),\qquad \exp\left(\int\left(1\,\mathrm{d}x_1+\frac{1}{x_2^2}\,\mathrm{d}x_2-\frac{1}{2x_3}\,\mathrm{d}x_3\right)\right),$$
$$\exp\left(\int\left(\frac{1}{x_1}\,\mathrm{d}x_1+2\,\mathrm{d}x_3\right)\right),\qquad \exp\left(\int\left(\frac{1}{x_1}\,\mathrm{d}x_1+\frac{1}{x_2^2}\,\mathrm{d}x_2+2\,\mathrm{d}x_3\right)\right),$$
$$\exp\left(\int 1\,\mathrm{d}x_1+2\,\mathrm{d}x_3\right),\qquad \exp\left(\int\left(\frac{1}{x_1}\,\mathrm{d}x_1+\frac{1}{x_2^2}\,\mathrm{d}x_2-\frac{1}{2x_3}\,\mathrm{d}x_3\right)\right).$$

Step 6 produces two genuine solutions: $\exp(x_1+2x_3)$ and $x_1/\sqrt{x_3}\exp(-1/x_2)$.

Example 3.4. Compute $\mathcal{H}(I)$, where the rank three ideal I is generated by

$$\begin{aligned}
&x_1v\partial_1 - x_2v\partial_2 + x_1(2x_1x_2 + x_1x_3 + x_2^2)\partial_3 - 2x_1x_2 - x_1x_3 + x_1^2,\\
&x_1(x_1-x_3)(2x_2-x_1+2x_3)\partial_3 + (x_2+x_3)^2v\partial_2^2 + 2x_1v\partial_2\\
&\qquad +x_1(-2x_2+x_1-2x_3),\\
&(x_2+x_3)^2v\partial_2\partial_3 + x_1v\partial_2 + 2x_1(x_2+x_3)(x_1-x_3)\partial_3 - 2x_1(x_2+x_3),\\
&(x_2+x_3)^2(x_1-x_3)v\partial_3^2 + w\partial_3 - 2x_1x_2^2 - 2x_1^2x_2 - 2x_1x_2x_3 - x_1^3 - x_3x_1^2,
\end{aligned}$$

where $v = x_1^2 - x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2$ and

$$\begin{aligned}
w = {}& -2x_2^4 - 8x_3x_2^3 - 12x_2^2x_3^2 - 8x_2x_3^3 + 2x_1x_3^3 + 2x_1^3x_2\\
& - 3x_1^2x_3^2 - 4x_1^2x_2x_3 - 2x_3^4 + x_1^4 + 2x_1x_2x_3^2.
\end{aligned}$$

Steps 1 and 2 are standard. Step 3 yields

$$\begin{aligned}
&\mathcal{H}^{(1)}(I_1) = \mathcal{H}^{(1)}_{(\alpha_1,\{1\})} \cup \mathcal{H}^{(1)}_{(\alpha_2,\{1,x_1\})},\qquad \mathcal{H}^{(2)}(I_2) = \mathcal{H}^{(2)}_{(1,\{1\})} \cup \mathcal{H}^{(2)}_{(\alpha_2,\{1,x_2\})},\\
&\text{and}\qquad \mathcal{H}^{(3)}(I_3) = \mathcal{H}^{(3)}_{(\alpha_1,\{1\})} \cup \mathcal{H}^{(3)}_{(\alpha_2,\{1\})},
\end{aligned}$$

where $\alpha_1 = 1/(x_1-x_3)$ and $\alpha_2 = \exp(x_1/(x_2+x_3))$. Step 4 finds two common associates and step 5 sets up two solution candidates:

$$\frac{c}{x_1-x_3},\qquad \exp\left(\frac{x_1}{x_2+x_3}\right)(c_0+c_1x_1+c_2x_2+c_3x_1x_2),$$

where $c, c_0, \ldots, c_3$ are constants. The algorithm returns solutions

$$h_1 = \frac{c}{x_1-x_3},\qquad h_2 = \exp\left(\frac{x_1}{x_2+x_3}\right)(c_0+c_3x_1x_2),$$

where c, c_0 and c_3 are unspecified constants with $c \neq 0$ and $c_0c_3 \neq 0$.

4. Hyperexponential solutions of submodules with finite rank

Let L be a submodule with finite rank in $\mathbb{L}_m$. A solution of L in $\widetilde{\mathbb{K}}^m$ is said to be hyperexponential if it is not $\vec{0}$, and each of its components is either hyperexponential or equal to zero. We extend the algorithm HyperexponentialSolutions in Section 3.3 to compute hyperexponential solutions of L.

Let $L^{(k)}$ be the intersection of L and $\mathbb{D}y_k$, for $k = 1, \ldots, m$, which can be computed by Janet basis computation w.r.t. an elimination ordering. The submodule $L^{(k)}$ of $\mathbb{D}y_k$ is of finite rank because L is. Applying the algorithm HyperexponentialSolutions to $L^{(k)}$, we obtain all possible hyperexponential solution candidates for y_k, in which there may be a finite number of unspecified constants. For each k with $1 \le k \le m$, we substitute these candidates into a set of generators of L to obtain a system of linear homogeneous equations in these unspecified constants by Proposition 3.3. Solve this system and make sure that there is at least one nonzero component in a solution (vector).

Example 4.1. Compute $\mathcal{H}(L)$ where L is generated by

$$\begin{aligned}
&(x_1x_3 - x_1x_2)y_1 + (x_1x_2 - 1)\partial_2 y_1 + (1 - x_1x_3)\partial_3 y_1,\\
&(x_2x_3 - x_1x_2)y_1 + (x_1x_2 - 1)\partial_1 y_1 + (1 - x_2x_3)\partial_3 y_1,\\
&\partial_1 y_2 - x_2x_3y_1 - x_2x_3y_2, \qquad \partial_2 y_2 - x_1x_3y_1 - x_1x_3y_2,\\
&\partial_3^2 y_1 + (x_1x_2 + 1)\partial_3 y_1 + x_1x_2y_1, \qquad \partial_3 y_2 - x_1x_2y_1 - x_1x_2y_2.
\end{aligned}$$

A Janet basis computation yields

$$\begin{aligned}
L^{(1)} &= (\partial_1 y_1 - x_2x_3y_1, \partial_2 y_1 - x_1x_3y_1, \partial_3 y_1 - x_1x_2y_1),\\
L^{(2)} &= (x_2^2x_1^2y_2 - 2x_1x_2\partial_3 y_2 + \partial_3^2 y_2, x_1\partial_1 y_2 - x_3\partial_3 y_2, x_2\partial_2 y_2 - x_3\partial_3 y_2).
\end{aligned}$$

Applying the algorithm HyperexponentialSolutions to $L^{(1)}$ and $L^{(2)}$, we obtain $y_1 = c_1 h$ and $y_2 = (c_2 + c_3x_1x_2x_3)h$, respectively, where $h = \exp(x_1x_2x_3)$ and c_1, c_2, c_3 are unspecified constants. Substituting the candidates (y_1, y_2) into a set of generators of L, we find that $c_1 = c_3$. Thus $y_2 = (c_2 + c_1x_1x_2x_3)h$. By Proposition 3.4, $c_4 = c_2/c_1$ belongs to $\overline{\mathbb{Q}}$. So $\mathcal{H}(L)$ is equal to the union of $\{(0, c_2h) \mid c_2 \in \widetilde{\mathbb{C}}, c_2 \ne 0\}$ and

$$\{(c_1h, (c_1c_4 + c_1x_1x_2x_3)h) \mid c_1 \in \widetilde{\mathbb{C}}, c_1 \ne 0, c_4 \in \bar{\mathbb{Q}}\}.$$

5. Factoring submodules with finite rank

This section presents an algorithm for solving Problem F. The algorithm hinges on the algorithm HyperexponentialSolutions and extends Algorithm F by Li et al. (2002). As there are several unknown functions, notation and constructions will be more involved. Nonetheless, the idea alters little. This section is structured as follows. Section 5.1 presents some useful facts. Section 5.2 studies quotient systems. Sections 5.3 and 5.4 generalize the notions of Wronskian and associated systems, respectively. The idea and algorithm for factorization are given in Sections 5.5 and 5.6, respectively.

5.1. Some useful facts

First, we show how to find rank one factors.

Proposition 5.1. *The submodule L has a rank one factor if and only if L has a hyperexponential solution whose nonzero components are equivalent to each other.*

Proof. If $\vec{h}$ is a hyperexponential solution of L whose ith component h_i is nonzero, and the jth component, where $j = 1, \ldots, n$, and $j \neq i$, is equal to $r_j h_i$ for some $r_j \in \mathbb{K}$, then L has a rank one factor generated by

$$\begin{aligned} &y_1 - r_1 y_i, \ldots, y_{i-1} - r_{i-1} y_i, \\ &\partial_1 y_i - ((\partial_1 \log) h_i)) y_i, \ldots, \partial_n y_i - ((\partial_n \log) h_i)) y_i, \\ &y_{i+1} - r_{i+1} y_i, \ldots, y_m - r_m y_i. \end{aligned} \tag{19}$$

Conversely, any factor with rank one has only one parametric derivative y_i for some i with $1 \leq i \leq m$. It follows that the factor can be generated by generators in the form (19). This factor has a hyperexponential solution

$$(r_1 h_i, \ldots, r_{i-1} h_i, h_i, r_{i+1} h_i, \ldots, r_m h_i). \quad \square$$

Example 5.1. The submodule given in Example 4.1 has two families of rank one factors $(\partial_1 y_1 - x_3 x_2 y_1, \partial_2 y_1 - x_1 x_3 y_1, \partial_3 y_1 - x_1 x_2 y_1, y_2 - (c_4 + x_1 x_2 x_3) y_1)$, in which $c_4 \in \bar{\mathbb{Q}}$, and $(y_1,\ \partial_1 y_2 - x_3 x_2 y_2, \partial_2 y_2 - x_1 x_3 y_2, \partial_3 y_2 - x_1 x_2 y_2)$.

For a dth order linear ode w.r.t. ∂_1, its kth order right factors have leading derivative ∂_1^k. What is $\mathrm{lder}(F)$ if F is a factor of L? The next lemma tells us that there are only finitely many choices for $\mathrm{lder}(F)$.

Lemma 5.2. *If $L \subset F$, then*

$$\mathrm{lder}(F) \subset (\mathrm{lder}(L) \cup \mathrm{pder}(L)) \quad \textit{and} \quad \mathrm{pder}(F) \subset \mathrm{pder}(L).$$

Proof. If $\delta \in \mathrm{lder}(F)$ and $\delta \notin (\mathrm{lder}(L) \cup \mathrm{pder}(L))$, then δ can be reduced by some γ in $\mathrm{lder}(L)$. As L is a subset of F, γ can be reduced by some ξ in $\mathrm{lder}(F)$, which is not equal to δ. Thus δ can be reduced by ξ, contradicting to the fact that $\mathrm{lder}(F)$ is autoreduced. The second assertion follows from the same argument. $\square$

Remark 5.2. The structure of factors of L can be described by the Jordan–Hölder theorem. See Tsarev (2001) and Li et al. (2002) for more details.

We use exterior algebra notation to denote determinants. Let $\mathbb{E} = \widetilde{\mathbb{K}}^m$. Recall that the application of $\gamma = \theta y_i \in \Gamma$ to a vector $\vec{v}$ in $\mathbb{E}$ is the application of θ to the ith component of $\vec{v}$. The k-fold exterior product $\lambda = \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_k$ is understood as a mapping from $\mathbb{E}^k$ to $\widetilde{\mathbb{K}}$:

$$\lambda(\vec{z}) = \begin{vmatrix} \gamma_1 \vec{z}_1 & \gamma_1 \vec{z}_2 & \cdots & \gamma_1 \vec{z}_k \\ \gamma_2 \vec{z}_1 & \gamma_2 \vec{z}_2 & \cdots & \gamma_2 \vec{z}_k \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_k \vec{z}_1 & \gamma_k \vec{z}_2 & \cdots & \gamma_k \vec{z}_k \end{vmatrix} \qquad \text{for } \vec{z} = (\vec{z}_1, \ldots, \vec{z}_k) \in \mathbb{E}^k.$$

For example, let $\gamma_1 = \partial_1 y_1$, $\gamma_2 = \partial_2^2 y_3$, $\vec{z}_1 = (z_{11}, z_{12}, z_{13})$ and $\vec{z}_2 = (z_{21}, z_{22}, z_{23})$, where the z_{ij} belong to $\widetilde{\mathbb{K}}$. Then

$$(\gamma_1 \wedge \gamma_2)(\vec{z}_1, \vec{z}_2) = \begin{vmatrix} (\partial_1 y_1)\vec{z}_1 & (\partial_1 y_1)\vec{z}_2 \\ (\partial_2^2 y_3)\vec{z}_1 & (\partial_2^2 y_3)\vec{z}_2 \end{vmatrix} = \begin{vmatrix} \partial_1 z_{11} & \partial_1 z_{21} \\ \partial_2^2 z_{13} & \partial_2^2 z_{23} \end{vmatrix}.$$

Besides being multi-linear and anti-symmetric, we also have

$$\partial_i(a\lambda) = (\partial_i(a))\lambda + a\sum_{j=1}^{k}(\gamma_1 \wedge \cdots \wedge (\partial_i\gamma_j) \wedge \cdots \wedge \gamma_k), \tag{20}$$

where $a \in \mathbb{K}, i = 1, \ldots, n$. We regard any $\mathbb{K}$-linear combination of k-fold exterior products of elements of Γ as a multi-linear function from $\mathbb{E}^k$ to $\widetilde{\mathbb{K}}$. Clearly, a derivation operator can be applied to such a combination. For a subset S of Γ, we denote by $\Lambda^k(S)$ the $\mathbb{K}$-linear space generated by all the k-fold exterior products of the elements of S. The $\mathbb{K}$-linear space $\Lambda^k(\Gamma)$ is closed under Θ. For every $\lambda \in \Lambda^k(\Gamma)$, there exists $\lambda_L \in \Lambda^k$ (pder(L)) s.t.

$$\lambda(\vec{z}_1, \ldots, \vec{z}_k) = \lambda_L(\vec{z}_1, \ldots, \vec{z}_k) \qquad \text{for } \vec{z}_1, \ldots, \vec{z}_k \in \operatorname{sol}(L). \tag{21}$$

The exterior expression λ_L can be computed by replacing each derivative appearing in λ by its normal form w.r.t. the Janet basis for L. Eq. (21) is crucial in the rest of this paper.

5.2. *Quotients*

A factor F of L helps us to find a subspace of $\operatorname{sol}(L)$. Can we use F to describe all the solutions of L? An answer is to use the quotient of L w.r.t. F, which is introduced by Tsarev (2001) and refined by Li et al. (2002) for the case in which $n = 2$ and $m = 1$. The general construction given below is similar.

Let $F = (F_1, \ldots, F_q)$ be a factor of $L = (L_1, \ldots, L_p)$. Assume that $\{F_1, \ldots, F_q\}$ be the reduced Janet (Gröbner) basis for F. Then, for each i with $1 \le i \le p$,

$$L_i = \sum_{j=1}^{q} Q_{ij}(F_j), \qquad \text{for some } Q_{ij} \in \mathbb{D} \text{ with } 1 \le i \le p \text{ and } 1 \le j \le q, \tag{22}$$

where $Q_{ij}(F_j)$ means the application of Q_{ij} to F_j. Since $\{F_1, \ldots, F_q\}$ is a Janet basis, all Δ-polynomials

$$\left(\frac{\delta_a(F_a)}{f_a} - \frac{\delta_b(F_b)}{f_b}\right) = \sum_{j=1}^{q} P_{abj}\left(F_j\right), \qquad \text{for some } P_{abj} \in \mathbb{D}, \tag{23}$$

where δ_a, δ_b are the derivatives to form the Δ-polynomial of F_a and F_b, and f_a, f_b are the respective leading coefficients. Let $u_1, \ldots, u_q$ be differential indeterminates over $\mathbb{K}$ and denote by $\mathbb{U}_q$ the submodule generated by $u_1, \ldots, u_q$. The *quotient* of L w.r.t. F and w.r.t. the term-order $\prec$ is defined to be the submodule in $\mathbb{U}_q$ generated by

$$Q = \{Q_i, T_{ab} \mid 1 \le i \le p, 1 \le a < b \le q\} \subset \sum_{j=1}^{q} \mathbb{D}u_j,$$

where

$$Q_i = \sum_{j=1}^{q} Q_{ij}(u_j), \qquad T_{ab} = \left(\frac{\delta_a(u_a)}{f_a} - \frac{\delta_b(u_b)}{f_b}\right) - \sum_{j=1}^{q} P_{abj}(u_j).$$

Proposition 5.3. *Let* $\vec{y} = (y_1, \ldots, y_m)$ *and let* $G(\vec{y}, u_1, \ldots, u_q)$ *denote the linear differential system* $\{F_1(\vec{y}) = u_1, \ldots, F_q(\vec{y}) = u_q\}$. *Then we have*

(1) *if* $(v_1, \ldots, v_q)$ *is in* $\mathrm{sol}(Q)$, *then there exists* $\vec{v}_0$ *in* $\mathbb{E}$ *s.t.* $(\vec{v}_0, v_1, \ldots, v_q)$ *is in* $\mathrm{sol}(G)$, *so that* $\vec{v}_0$ *is in* $\mathrm{sol}(L)$*;*

(2) *if* $\vec{v}_0 \in \mathrm{sol}(L)$, *then* $(F_1(\vec{v}_0), \ldots, F_q(\vec{v}_0)) \in \mathrm{sol}(Q)$*;*

(3) $\dim \mathrm{sol}(Q) + \dim \mathrm{sol}(F) = \dim \mathrm{sol}(L)$.

Proof. We begin to prove the first assertion. Let F be a factor of rank k and $(v_1, \ldots, v_q)$ belongs to $\mathrm{sol}(Q)$. Regard $G(\vec{y}) = G(\vec{y}, v_1, \ldots, v_q)$ as a differential system in $y_1, \ldots, y_m$. Its integrability conditions, i.e. Δ-polynomials $T_{ab}(v_1, \ldots, v_q)$ $(1 \leq a < b \leq q)$, vanish, since all the T_{ab} are in Q. In other words, $\{F_1(\vec{y}) - v_1, \ldots, F_q(\vec{y}) - v_q\}$ is a linear coherent autoreduced set. Hence, $G(\vec{y})$ has a solution $\vec{y}_0$ in $\mathbb{E}$. It follows from (22) that $\vec{y}_0 \in \mathrm{sol}(L)$. The second assertion is direct from (22) and (23).

To prove the last assertion, we recall that $d = \dim \mathrm{sol}(L)$. Let h be the dimension of $\mathrm{sol}(Q)$ over $\widetilde{\mathbb{C}}$, and $\vec{z}_1, \ldots, \vec{z}_k, \vec{w}_1, \ldots, \vec{w}_{d-k}$ form a basis of $\mathrm{sol}(L)$, in which $\vec{z}_1, \ldots, \vec{z}_k$ are in $\mathrm{sol}(F)$. Then the vectors $\vec{r}_i = (F_1(\vec{w}_i), \ldots, F_q(\vec{w}_i))$, where $1 \leq i \leq (d-k)$, are nontrivial solutions of Q by the second assertion. If $\vec{r}_1, \ldots, \vec{r}_{d-k}$ are $\widetilde{\mathbb{C}}$-linearly dependent, then a nontrivial $\widetilde{\mathbb{C}}$-linear combination of the $\vec{w}_i$ is a solution of all the F_i, a contradiction to the selection of the $\vec{w}_i$. Thus, $h \geq (d-k)$. For nonzero $\vec{v} \in \mathrm{sol}(Q)$, there is a solution $\vec{y}_0$ of $G(\vec{y}, \vec{v})$ by the first assertion. Since $\vec{y}_0 \in \mathrm{sol}(L)$, it can be expressed as a nontrivial $\widetilde{\mathbb{C}}$-linear combination of the $\vec{z}_j$ and $\vec{w}_i$. Applying each F_l to the linear combination, we see that $\vec{v}$ is a $\widetilde{\mathbb{C}}$-linear combination of $\vec{r}_1, \ldots, \vec{r}_{d-k}$. Consequently, we get $h \leq (d-k)$. □

Example 5.3. Let us consider the submodule L given in Example 4.1. Example 5.1 shows that L has a factor F generated by

$$f_1 = \partial_1 y_2 - x_2 x_3 y_2, \qquad f_2 = \partial_2 y_2 - x_1 x_3 y_2,$$
$$f_3 = \partial_3 y_2 - x_1 x_3 y_2, \qquad f_4 = y_1.$$

A quotient Q of L and F is generated by nine elements, six of which correspond to the reduction of generators of L by the f's (see (22)), and three of which correspond to Δ-polynomials among the f's (see (23)). Using these nine elements, we compute a Janet basis to get Q equal to

$$(u_1 - x_2 x_3 u_4, u_2 - x_1 x_3 u_4, u_3 - x_1 x_2 u_4,$$
$$\partial_1 u_4 - x_2 x_3 u_4, \partial_2 u_4 - x_1 x_3 u_4, \partial_3 u_4 - x_1 x_2 u_4).$$

It has a solution $(x_2 x_3 h, x_1 x_3 h, x_1 x_2 h, h)$, where $h = \exp(x_1 x_2 x_3)$. The first assertion of Proposition 5.3 prompts us to form the system G equal to

$$\{\partial_1 u_{02} - x_2 x_3 u_{02} = x_2 x_3 h,\ \partial_2 u_{02} - x_1 x_3 u_{02} = x_1 x_3 h,$$
$$\partial_3 u_{02} - x_1 x_2 u_{02} = x_1 x_2 h, u_{01} = h\}.$$

By variation of parameters we find that $(h, x_1 x_2 x_3 h)$ solves G. Hence, it is a solution of L by Proposition 5.3. A basis for $\mathrm{sol}(L)$ is $\{(0, h), (h, x_1 x_2 x_3 h)\}$.

5.3. Wronskian representations

A key idea in the Beke–Schlesinger algorithm is to look for right factors whose coefficients are Wronskian-like determinants. To use this idea, we extend the notion of Wronskians. Let F be a submodule with finite rank k. The reduced (monic) Janet basis for F consists of $\{F_1, \ldots, F_p\}$. Let $\mathrm{lder}(F) = \{\gamma_1, \ldots, \gamma_p\}$ and $\mathrm{pder}(F) = \{\xi_1, \ldots, \xi_k\}$, where $\xi_i \prec \xi_j$ for $1 \le i < j \le k$.

We call the element $\omega_F = (\xi_1 \wedge \cdots \wedge \xi_k)$ the *Wronskian operator* of F (w.r.t. the term order $\prec$). It follows from (21) and $\Lambda^k(\mathrm{pder}(F)) = \{r\omega_F \mid r \in \mathbb{K}\}$ that, for every $\lambda \in \Lambda^k(\Gamma)$, there exists $r_\lambda \in \mathbb{K}$ such that

$$\lambda(\vec{z}_1, \ldots, \vec{z}_k) = r_\lambda \omega_F(\vec{z}_1, \ldots, \vec{z}_k) \qquad \text{for } \vec{z}_1, \ldots, \vec{z}_k \in \mathrm{sol}(F). \tag{24}$$

Lemma 5.4. *For all $\vec{z}_1, \ldots, \vec{z}_k \in \mathrm{sol}(F)$, $\vec{z}_1, \ldots, \vec{z}_k$ are $\widetilde{\mathbb{C}}$-linearly independent if and only if $\omega_F(\vec{z}_1, \ldots, \vec{z}_k) \neq 0$. Moreover, let $\vec{z}_1, \ldots, \vec{z}_k$ form a basis of $\mathrm{sol}(F)$ and denote $(\vec{z}_1, \ldots, \vec{z}_k)$ by $\vec{z}$, $(y_1, \ldots, y_m)$ by $\vec{y}$. Then*

$$(\omega_F \wedge \gamma_i)(\vec{z}, \vec{y}) = \omega_F(\vec{z}) F_i, \qquad i = 1, \ldots, p.$$

Proof. If $\vec{z}_1, \ldots, \vec{z}_k$ are $\widetilde{\mathbb{C}}$-linearly independent, Theorem 1 in Kolchin (1973, p. 86) implies that there exists λ in $\Lambda^k(\Gamma)$ s.t. $\lambda(\vec{z}_1, \ldots, \vec{z}_k) \neq 0$. The first assertion then follows from (24). The converse is true by the same theorem. Expanding $(\omega_F \wedge \gamma_i)(\vec{z}, \vec{y})$ according to the last column, we have

$$(\omega_F \wedge \gamma_i)(\vec{z}, \vec{y}) = \omega_F(\vec{z})\gamma_i + \sum_{j=1}^{k} \underbrace{(-1)^{k+j+1}(\eta_j \wedge \gamma_i)(\vec{z})}_{w_{ij}} \xi_j, \tag{25}$$

where $\eta_j = \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_k$. Since $(\omega_F \wedge \gamma_i)(\vec{z}, \vec{y})$ vanishes on $\mathrm{sol}(F)$, it can be reduced to zero by $\{F_1, \ldots, F_p\}$. But the right-hand side of (25) can only be reduced by F_i once. The second assertion is proved. $\square$

We call $\{(\omega_F \wedge \gamma_1)(\vec{z}, \vec{y}), \ldots, (\omega_F \wedge \gamma_p)(\vec{z}, \vec{y})\}$ a *Wronskian representation* of F. Any two Wronskian representations can only differ by a multiplicative constant in $\widetilde{\mathbb{C}}$, because any two sets of fundamental solutions of F can be transformed from one to the other by a matrix over $\widetilde{\mathbb{C}}$. Note that $w_{ij} = 0$ if $\gamma_i \prec \xi_j$, because of the second assertion of Lemma 5.4.

Example 5.4. Let $\mathrm{lder}(F) = \{y_1, \partial_1 y_2, \partial_2 y_2, \partial_3^2 y_2\}$. Then $\mathrm{pder}(F) = \{y_2, \partial_3 y_2\}$. The Wronskian operator is $\omega_F = (y_2 \wedge (\partial_3 y_2))$ and the representation is

$$\begin{aligned}
\{W_1 &= \omega_F(\vec{z})y_1 - (y_2 \wedge y_1)(\vec{z})(\partial_3 y_2) + ((\partial_3 y_2) \wedge y_1)y_2,\\
W_2 &= \omega_F(\vec{z})\partial_1 y_2 - (y_2 \wedge (\partial_1 y_2))(\vec{z})(\partial_3 y_2) + ((\partial_3 y_2) \wedge (\partial_1 y_2))y_2,\\
W_3 &= \omega_F(\vec{z})\partial_2 y_2 - (y_2 \wedge (\partial_2 y_2))(\vec{z})(\partial_3 y_2) + ((\partial_3 y_2) \wedge (\partial_1 y_2))y_2,\\
W_4 &= \omega_F(\vec{z})\partial_3^2 y_2 - (y_2 \wedge (\partial_3^2 y_2))(\vec{z})(\partial_3 y_2) + ((\partial_3 y_2) \wedge (\partial_3^2 y_2))y_2\}.
\end{aligned}$$

The next proposition implies that the w_{ij} in (25) is hyperexponential.

Proposition 5.5. *Let F be a submodule of rank k in $\mathbb{L}_m$, and $\vec{z} = (\vec{z}_1, \ldots, \vec{z}_k)$, where the $\vec{z}_i$ form a basis of $\mathrm{sol}(F)$. Then $\omega_F(\vec{z})$ is hyperexponential over $\mathbb{K}$. Moreover, for all $\lambda \in \Lambda^k(\Gamma)$, $\lambda(\vec{z})$ is either zero or hyperexponential.*

Proof. Lemma 5.4 implies that $\omega_F(\vec{z})$ is nonzero. It follows from (24) that the logarithmic derivative of $\omega_F(\vec{z})$ w.r.t. any x_i belongs to $\mathbb{K}$. Hence, $\omega_F(\vec{z})$ is hyperexponential. Any nonzero $\lambda(\vec{z})$ is then hyperexponential by (24). □

5.4. Associated systems

We shall generalize the notion of associated equations for factoring linear ODEs. As in the previous sections, let k be an integer with $1 \le k < d$. We regard every element of $\Lambda^k(\Gamma)$ as a function on $\mathrm{sol}(L)^k$. Two elements of $\Lambda^k(\Gamma)$ are said to be *equivalent* if they are identical (as functions) on $\mathrm{sol}(L)^k$. For an element λ of $\Lambda^k(\Gamma)$, its equivalence class is denoted by $\bar{\lambda}$. It is easy to verify that the equivalence relation is compatible with linear operations and differentiations on $\Lambda^k(\Gamma)$. The $\mathbb{K}$-linear space consisting of the equivalence classes is called the kth *Beke space* relative to L, and denoted by B_k when L is clear from the context. From (21) it follows that each equivalence class contains an element of $\Lambda^k(\mathrm{pder}(L))$. Consequently, B_k can be $\mathbb{K}$-linearly generated by the elements in the form $\overline{(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_k)}$, where the η_i belong to $\mathrm{pder}(L)$ and $\eta_i \prec \eta_j$ for all i, j with $1 \le i < j \le k$. These elements are called *canonical generators* of B_k. They are not necessarily $\mathbb{K}$-linearly independent.

Lemma 5.6. *The kth Beke space B_k is of dimension less than or equal to $\binom{d}{k}$ and closed under differentiation.*

Example 5.5. Let $\mathrm{pder}(L) = \{y_1, y_2, \partial_3 y_2\}$. The canonical generators of the second Beke's space B_2 are $b_1 = \overline{(y_1 \wedge y_2)}$, $b_2 = \overline{(y_1 \wedge (\partial_3 y_2))}$, $b_3 = \overline{(y_2 \wedge (\partial_3 y_2))}$.

Set $e = \binom{d}{k}$. For an element $\bar{\lambda}$ of B_k, the ideal consisting of all annihilators of $\bar{\lambda}$ in $\mathbb{D}$ is denoted by $\mathrm{ann}(\bar{\lambda})$. A finite subset of $\mathrm{ann}(\bar{\lambda})$ with finite-dimensional solution space is called a *system associated with* $\bar{\lambda}$. The following method computes an associated system by linear algebra and differential reduction. Lemma 5.6 implies that $\bar{\lambda}, \partial_i \bar{\lambda}, \ldots, \partial_i^e \bar{\lambda}$ are linearly dependent over $\mathbb{K}$. Suppose that p_i is a smallest nonnegative integer such that $\partial_i^{p_i} \bar{\lambda} + \sum_{j=0}^{p_i-1} f_{ij} \partial_i^j \bar{\lambda} = 0$, where $f_{i,p_i-1}, \ldots, f_{i0} \in \mathbb{K}$. We find the ideal generated by

$$\left\{ \partial_i^{p_i} + \sum_{j=0}^{p_i-1} f_{ij} \partial_i^j \mid i = 1, \ldots, n \right\} \tag{26}$$

annihilating $\bar{\lambda}$. The solution space of the ideal is of finite dimension, because its parametric derivatives are in $D_\lambda = \{\partial_1^{i_1} \cdots \partial_n^{i_n} \mid 0 \le i_j \le p_j - 1, 1 \le j \le n\}$. Hence, (26) is a system associated with $\bar{\lambda}$. Considering all possible $\mathbb{K}$-linear combinations of $(e+1)$ elements of D_λ, we may obtain an associated system with e-dimensional solution space (see Lemma 1 in Tsarev, 2001). We may also consider linear relations among mixed derivatives of $\bar{\lambda}$ to get associated systems with lower-dimensional solution space.

To factor submodules with finite rank, we need systems associated with the canonical generators. The method for computing these associated equations given in Bronstein (1994) can be directly applied in the general case.

Example 5.6. Consider the rank three submodule L generated by

$$
\begin{aligned}
L_1 &= x_1^2x_2(\partial_1 y_1) + x_3x_1^2x_2^2y_2 - (\partial_3 y_3)x_3x_2x_1 - (\partial_3 y_3) + x_1x_2y_1 + y_2x_1x_2,\\
L_2 &= x_1x_2^2(\partial_2 y_1) + x_3x_1^2x_2^2y_2 - (\partial_3 y_3)x_3x_2x_1 - (\partial_3 y_3) + x_1x_2y_1 + y_2x_1x_2,\\
L_3 &= (\partial_3 y_1) + y_2x_1x_2 - (\partial_3 y_3), \qquad L_4 = (\partial_3^2 y_3) + x_1^2x_2^2y_2 - 2x_1x_2(\partial_3 y_3),\\
L_5 &= (\partial_1 y_2) - (\partial_3 y_3)x_3, \qquad L_6 = x_2(\partial_2 y_2) - (\partial_3 y_3)x_3.
\end{aligned}
$$

The set $\{L_1, \ldots, L_6\}$ is a reduced Janet basis under the lexicographical term-order defined by $y_1 < y_2 < y_3$ and $\partial_1 < \partial_2 < \partial_3$. Thus,

$$\mathrm{lder}(F) = \{\partial_1 y_1, \partial_2 y_2, \partial_3 y_3, \partial_3^2 y_2, \partial_2 y_2, \partial_1 y_2\}, \quad \text{and} \quad \mathrm{pder}(F) = \{y_2, y_1, \partial_3 y_2\}.$$

The canonical generators of B_2 are b_1, b_2 and b_3 as given in Example 5.5. By linear algebra and differentiation, we find ideals I_1, I_2, I_3 annihilating b_1, b_2 and b_3, respectively, where $I_1 = (\partial_1 - \frac{2t+1}{x_1}, \partial_2 - \frac{2t+1}{x_2}, \partial_3 - 2x_1x_2)$,

$$
\begin{aligned}
I_2 = \Bigg(&\frac{-2x_2^2x_3^2(t^2-6)}{x_1(t+2)} + \frac{t(-18+4t+5t^2)}{x_1^2(t+2)}\partial_1\\
&- \frac{2(3t-3+2t^2)}{x_1(t+2)}\partial_1^2 + \partial_1^3, \frac{-2t(t^2-6)}{x_2^3(t+2)} + \frac{t(-18+4t+5t^2)}{x_2^2(t+2)}\partial_2\\
&- \frac{2(3t-3+2t^2)}{x_2(t+2)}\partial_2^2 + \partial_2^3, -2x_1^3x_2^3 + 5x_1^2x_2^2\partial_3 - 4x_1x_2\partial_3^2 + \partial_3^3\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 = \Bigg(&\frac{-t^3(2t+5)}{x_1^3(t+2)} + \frac{t^2(5t+12)}{x_1^2(t+2)}\partial_1 - \frac{t(4t+9)}{x_1(t+2)}\partial_1^2 + \partial_1^3, \frac{-t^3(2t+5)}{x_2^3(t+2)}\\
&+ \frac{t^2(5t+12)}{x_2^2(t+2)}\partial_2 - \frac{t(4t+9)}{x_2(t+2)}\partial_2^2 + \partial_2^3, -2x_1^3x_2^3 + 5x_1^2x_2^2\partial_3 - 4x_1x_2\partial_3^2 + \partial_3^3\Bigg),
\end{aligned}
$$

in which $t = x_1x_2x_3$.

5.5. *Sketch of the factorization algorithm*

Before presenting our factorization algorithm in detail, we describe it informally by examples. Assume that we look for a rank k factor $F \subset \mathbb{L}_m$ of L. Let $\vec{z}$ the vector $(\vec{z}_1, \ldots, \vec{z}_k)$ where $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_k$ form a fundamental system of solutions of F.

First, we enumerate all possible leading derivatives of F by Lemma 5.2.

Example 5.7. The submodule L given in Example 5.6 might have rank two factors whose leading derivatives are

$$\{y_1, \partial_1 y_2, \partial_2 y_2, \partial_3^2 y_3\} \qquad \text{or} \qquad \{\partial_1 y_2, \partial_2 y_2, \partial_3 y_3, \partial_1 y_1, \partial_2 y_1, \partial_3 y_1\}.$$

Second, for a given $\mathrm{lder}(F)$, we compute candidates for the Wronskian operator $\omega_F(\vec{z})$ by finding the hyperexponential solutions of one of its associated systems. If no hyperexponential solution is found, then the factor with the leading derivatives $\mathrm{lder}(F)$ does not exist by Proposition 5.5.

Example 5.8. Find a factor F of L with $\mathrm{lder}(F) = \{y_1, \partial_1 y_2, \partial_2 y_2, \partial_3^2 y_3\}$. The Wronskian operator of F is $\omega_F = \overline{y_2 \wedge (\partial_3 y_2)}$. An ideal annihilating ω_F is I_1 in Example 5.6. The algorithm HyperexponentialSolutions finds that $\omega_F(\vec{z})$ can only be $c_0 x_1 x_2 h^2$, where $h = \exp(x_1 x_2 x_3)$ and c_0 is a constant.

Third, we compute all candidates for canonical generators equivalent to a given Wronskian candidate, because (24) implies that all candidates for canonical generators are hyperexponential and equivalent to the Wronskian candidate. If the Wronskian candidate is equivalent to h, then candidates for a canonical generator b can be expressed as rh where r belongs to $\mathbb{K}$. Substituting rh into a system associated with b, we obtain an ideal with finite rank. We are only interested in its rational solutions.

Example 5.9. Besides the Wronskian operator ω_F, two other canonical generators of the second Beke space are $b_2 = \overline{y_2 \wedge y_1}$ and $b_3 = \overline{(\partial_3 y_2) \wedge y_1}$, $b_2(\vec{z})$ is annihilated by I_2, $b_3(\vec{z})$ by I_3, as described in Example 5.6. The hyperexponential solutions of I_2 (resp. I_3) equivalent to h^2 are $c_1 h^2$ (resp. $c_2 x_1 x_2 h$), where c_1 and c_2 are constants.

Fourth, we form all the Wronskian representations for F w.r.t. a given candidate for the Wronskian operator. This is possible because the coefficients w_{ij} in (25) are $\mathbb{K}$-linear combinations of the $b_i(\vec{z})$ which can be obtained by the reduction w.r.t. L.

Example 5.10. The Wronskian representation of F is given in Example 5.4. All its coefficients are $\mathbb{K}$-linear combinations of $\omega_F(\vec{z})$, $b_2(\vec{z})$ and $b_3(\vec{z})$. These combinations can be found by the reduction of an element B_2 by L. The Wronskian representation of F is

$$\begin{aligned}&\{x_1x_2y_1 - c_1(\partial_3 y_2) + x_1x_2c_2y_2,\ x_1x_2(\partial_1 y_2) - x_3x_2(\partial_3 y_2),\\ &x_1x_2(\partial_3^2 y_3) - 2x_1^2x_2^2(\partial_3 y_2) + x_1^3x_2^3 y_2,\ x_1x_2(\partial_2 y_2) - x_3x_1(\partial_3 y_2)\}.\end{aligned} \tag{27}$$

Fifth, the monic associate U of a candidate for the Wronskian representation has rational coefficients. If the monic associate is a factor of L with rank k, then U is a reduced Janet basis and each element of L can be reduced to zero by U. These two constraints lead to a system of algebraic equations in the unspecified constants appearing in U. Solving these algebraic equations yields factors that we seek.

Example 5.11. Decide the constants in (27) by assuming that the monic associate of (27) is a Janet basis and that L is contained in F. We get a factor F generated by $y_1 - 1/x_1x_2(\partial_3 y_2) + y_2$, $(\partial_1 y_2) - x_3/x_1(\partial_3 y_2)$, $(\partial_2 y_2) - x_3/x_2(\partial_3 y_2)$, and $(\partial_3^2 y_3) - 2x_1x_2(\partial_3 y_2) + x_1^2x_2^2 y_2$.

5.6. Factorization algorithm

For simplicity, we describe an algorithm for finding factors F of L under the assumption that $\mathrm{lder}(F)$ is given. It is easy to adjust the algorithm to compute all factors of L by Lemma 5.2.

FactorWithSpecifiedLeaders (Compute Factors whose Leaders are given). Given a finite basis for a submodule L of finite rank and an autoreduced set Δ in the union of $\mathrm{lder}(L)$ and $\mathrm{pder}(L)$, the algorithm computes all proper factors F of L with $\mathrm{lder}(F) = \Delta$.

1. [Parametric derivatives]. Find $\Delta^- \subset \mathrm{pder}(L)$ consisting of all derivatives not divisible by any elements of Δ. If $|\Delta^-| = d$, return L. Otherwise, set $k = |\Delta^-|$ and $e = \binom{d}{k}$.
2. [Candidates for the Wronskian]. Find a system A_1 associated with ω_F, and compute hyperexponential solutions of A_1 by HyperexponentialSolutions. If no hyperexponential solution is found, exit [no such factors exist]. Otherwise, organize the solutions as equivalence classes:
$$\{h_{11} = p_{11} f_{11}, \ldots, h_{1t} = p_{1t} f_{1t}\}$$
where the f_{1i} are hyperexponential over $\mathbb{K}$, and the p_{1i} are polynomials in $x_1, \ldots, x_n$ whose coefficients are elements of $\bar{\mathbb{Q}}$ and unspecified constants.
3. [Candidates for other canonical generators]. Construct the systems $A_2, \ldots, A_e$ associated with other canonical generators, and compute their hyperexponential solutions equivalent to some f_{1i} $(1 \le i \le t)$. For $j = 2, \ldots, e$, set h_{ji} to be the hyperexponential solution of A_j equivalent to f_{1i} if such a solution exists, else set h_{ji} to be zero. Let
$$H = \{(h_{11}, h_{21}, \ldots, h_{e1}), \ldots, (h_{1t}, h_{2k}, \ldots, h_{et})\}$$
where the h_{1i} are obtained from step 2, and the h_{ji} with $j > 1$ are either zero or hyperexponential elements equivalent to h_{1i}.
4. [Candidates for factors]. Construct the Wronskian representation defined by Δ. Construct the matrix transforming the canonical generators to the Wronskian coefficients. Use this matrix and the elements of H to get all rational monic associates $\{F_1, \ldots, F_{|\Delta|}\}$ of the candidates for factors.
5. [Select true factors]. Check if each F_i is reduced Janet basis and if F_i contains L. Solve algebraic equations in unspecified constants when necessary. Return the factors.

A few words need to be said about FactorWithSpecifiedLeaders. The first step is clear. The second step is a direct application of the algorithm HyperexponentialSolutions. If no hyperexponential solution is found, then factors with leading derivatives Δ do not exist by Proposition 5.5. In the third step, (24) implies that we need only hyperexponential solutions equivalent to some h_{1i}. Since these solutions belong to one equivalence class, all of them can be expressed as $q_i h_{1i}$, where q_i is a rational function whose coefficients are elements of $\bar{\mathbb{Q}}$ and unspecified constants. Thus, H contains at most t elements. Finding these solutions amounts to computing rational solutions of some ideals with finite rank, which is easier than computing all hyperexponential solutions of other associated systems. This technique is introduced by Bronstein (1994) for the ODE case, and is extended to the PDE case by Tsarev (2001). In the fourth step, we express the Wronskian coefficients as $\mathbb{K}$-linear combinations of the canonical generators by differential reduction w.r.t. L. In the last step there may arise an algebraic system in unspecified constants. So an algebraic equation solver is required.

Example 5.12. Let us find factors G of L (in Example 5.6) whose leading derivatives are $\{\partial_1 y_2, \partial_2 y_2, \partial_3 y_3, \partial_1 y_1, \partial_2 y_1, \partial_3 y_1\}$. Since $\mathrm{pder}(G) = \{y_2, y_1\}$, the Wronskian operator ω_G is equal to $b_2 = \overline{y_2 \wedge y_1}$. The algorithm HyperexponentialSolutions finds $b_2(\vec{z})$ can only be β_{21} equal to

$$(c_1 + c_2x_3 + c_3x_1x_2 + c_4x_1x_2x_3 + c_5x_1 + c_6x_1x_3 + c_7x_2 + c_8x_2x_3)\frac{h}{x_1x_2}$$

or $\beta_{22} = c_0h^2$, where $h = \exp(x_1x_2x_3)$ and the c's are constants. Other two canonical generators of B_2 are $b_1 = \overline{y_2 \wedge (\partial_3 y_2)}$ and $b_3 = \overline{(\partial_3 y_2) \wedge y_1}$. The hyperexponential solutions of I_1 and I_3 equivalent to β_{21} are 0 and

$$\begin{aligned}\beta_{31} = h(&c_9 + c_{10}x_3 + x_1x_2c_{11} + x_1x_2c_{12}x_3\\ &+ x_1c_{13} + x_1c_{14}x_3 + x_2c_{15} + x_2c_{16}x_3).\end{aligned}$$

The respective hyperexponential solutions of I_1 and I_3 equivalent to β_{22} are

$$\beta_{12} = c_0x_1x_2h^2 \qquad \text{and} \qquad \beta_{32} = c_1x_1x_2h^2.$$

The Wronskian representation for G by definition is

$$\begin{aligned}W_G = \{&b_2(\vec{z})(\partial_1 y_1) - \overline{y_2 \wedge (\partial_1 y_1)}(\vec{z})y_1 + \overline{y_1 \wedge (\partial_1 y_1)}(\vec{z})y_2, b_2(\vec{z})(\partial_2 y_1)\\ &- \overline{y_2 \wedge (\partial_2 y_1)}(\vec{z})y_1 + \overline{y_1 \wedge (\partial_2 y_2)}(\vec{z})y_2, b_2(\vec{z})(\partial_3 y_1) - \overline{y_2 \wedge (\partial_3 y_1)}(\vec{z})y_1\\ &+ \overline{y_1 \wedge (\partial_3 y_1)}(\vec{z})y_2, b_2(\vec{z})(\partial_1 y_2) - \overline{y_1 \wedge (\partial_1 y_2)}(\vec{z})y_2, b_2(\vec{z})(\partial_2 y_2)\\ &- \overline{y_1 \wedge (\partial_2 y_2)}(\vec{z})y_2, b_2(\vec{z})(\partial_3 y_2) - \overline{y_1 \wedge (\partial_3 y_2)}(\vec{z})y_2\}.\end{aligned}$$

Note that

$$\overline{y_2 \wedge (\partial_1 y_2)} = \overline{y_2 \wedge (\partial_2 y_2)} = \overline{y_2 \wedge (\partial_3 y_2)} = 0$$

since $\partial_3 y_2 < \partial_2 y_2 < \partial_1 y_2 < y_1$. Hence, $b_1(\vec{z}) = 0$. Consequently $\beta_{12} = 0$. We have two candidates for the canonical generators, which are

$$(b_1, b_2, b_3) = (0, \beta_{21}, \beta_{31}), \qquad (b_1, b_2, b_3) = (0, \beta_{22}, \beta_{32}).$$

The first candidate leads to a factor

$$\begin{aligned}&(x_1s(\partial_1 y_1) + sy_1 - c_4(t+1)y_2, x_2s(\partial_2 y_1) + sy_1 + c_4(t+1)y_2,\\ &s(\partial_3 y_1) - c_4x_1x_2y_2, s(\partial_1 y_2) - c_4x_2x_3ty_2 - c_9x_2x_3y_2,\\ &s(\partial_2 y_2) - c_4x_1x_3ty_2 - c_9x_1x_3y_2, s(\partial_3 y_2) - c_4x_1x_2ty_2 - c_9x_1x_2y_2),\end{aligned}$$

where $t = x_1x_2x_3$ and $s = (t-1)c_4 + c_9$. The second leads to a factor which is a special instance of the first ($c_4 = 0, c_9 = 1$).

The reader is referred to Li et al. (2002) for examples on factorization in $\mathbb{D}$.

6. Concluding remarks

The results of this article are a first step toward generalizing computer algebra techniques for solving linear ODEs to PDE's. The algorithm HyperexponentialSolutions generalizes the algorithm for computing hyperexponential solutions of linear ODEs.

The algorithm FactorWithSpecifiedLeaders generalizes the Beke–Schlesinger algorithm for factoring linear ODEs. The notions of factors and quotients enable us to reduce the rank of a D-finite system.

Based on the Maple packages `Ore_algebra` and `DEtools`, a preliminary implementation of the algorithm HyperexponentialSolutions has been made. The factorization algorithm for ideals in $\bar{\mathbb{Q}}(x_1, x_2)[\partial_1, \partial_2]$ has been implemented in the ALLTYPES system of Schwarz (1998). Yet, it is challenging to have an efficient factorizer for D-finite systems with rational function coefficients. To this end, we would like to have efficient implementations for finding elimination ideals I_i in Section 3, and computing the solutions of I_i in $\mathbb{K}$. We will study how to avoid generating too many candidates for hyperexponential solutions in HyperexponentialSolutions and how to construct A_1 in step 2 of FactorWithSpecifiedLeaders with lower rank so that we may have fewer candidates for factors in step 4. To factor a D-finite system, we would have to enumerate all possible sets of leading derivatives of a potential factor. The number of these sets may be an exponential function in rank. Would there be a fast way to decide if a set of leading derivatives will not lead to any true factor? Would there be a fast way to decide if an ideal with finite rank has no hyperexponential solutions?

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References

Beke, E., 1894. Die Irreduzibilität der homogenen linearen Differentialgleichungen. Math. Ann. 45, 278–294.

Borel, A., Grivel, P.P., Kaup, B., Haefliger, A., Malgrange, B., Ehlers, F., 1987. Algebraic D-Modules. Academic Press, Boston.

Boulier, F., Lazard, D., Ollivier, F., Petitot, M., 1995. Representation for the radical of a finitely generated differential ideal. In: Levelt, A.H.M. (Ed.), ISSAC'95, Proc. Int. Symp. on Symbolic and Algebraic Computation. ACM Press, Montreal, Canada, pp. 158–166.

Bronstein, M., 1992. Linear ordinary differential equations: breaking through the order 2 barrier. In: Wang, P. (Ed.), ISSAC'92, Proc. Int. Symp. on Symbolic and Algebraic Computation. ACM Press, Berkeley, pp. 42–48.

Bronstein, M., 1994. An improved algorithm for factoring linear ordinary differential operators. In: Giesbrecht, M. (Ed.), ISSAC'94, Proc. Int. Symp. on Symbolic and Algebraic Computation. ACM Press, Oxford, pp. 336–340.

Bronstein, M., 1997. Symbolic Integration I: Transcendental Functions. Springer-Verlag.

Chyzak, F., 2000. An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math. 217, 115–134.

Chyzak, F., Salvy, B., 1998. Non-commutative elimination in Ore algebras proves multivariate identities. J. Symbolic Comput. 26, 187–227.

Galligo, A., 1985. Some algorithmic questions on ideals of differential operators. In: Buchberger B. (Ed.) EUROCAL' 85. Springer Lecture Notes in Computer Science, vol. 204. pp. 413–421.

Geddes, K., Czapor, S., Labahn, G., 1992. Algorithms for Computer Algebra. Kluwer Academic Publishers.

Janet, M., 1920. Les systéms d'équations aux dérivées partielles. J. de mathématiques 83, 65–123.

Kandru-Rody, A., Weispfenning, V., 1990. Non-commutative Gröbner bases in algebras of solvable type. J. Symbolic Comput. 9, 1–26.

Kolchin, E., 1973. Differential Algebra and Algebraic Groups. Academic Press, New York.

Li, Z., Schwarz, F., 2001. Rational solutions of Riccati-like partial differential equations. J. Symbolic Comput. 31, 691–716.

Li, Z., Schwarz, F., Tsarev, S.P., 2002. Factoring zero-dimensional ideals of linear partial differential operators. In: Mora, T. (Ed.), ISSAC'02, Proc. Int. Symp. on Symbolic and Algebraic Computation. ACM Press, Lille, France, pp. 168–175.

Lie, S. Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten III. Archiv for Mathematik VIII, pp. 371–458 (Gesammelte Abhandlungen V, pp. 362–427).

Oaku, T., Takayama, N., Tsai, H., 2001. Polynomial and rational solutions of holonomic systems. J. Pure Appl. Algebra 164, 199–220.

Rosenfeld, A., 1959. Specializations in differential algebra. Trans. Amer. Math. Soc 90, 394–407.

Saito, M., Sturmfels, B., Takayama, N., 2000. Gröbner Deformations of Hypergeometric Differential Equations. Springer-Verlag.

Schlesinger, L., 1895. Handbuch der Theorie der linearen Differentialgleichungen. Teubner, Leipzig.

Schwarz, F., 1989. A factorization algorithm for linear ordinary differential equations. In: Gonnet, G. (Ed.), ISSAC'89, Proc. Int. Symp. on Symbolic Algebraic Comput. ACM Press, Protland, pp. 17–25.

Schwarz, F., 1992. Reduction and completion algorithms for partial differential equations. In: Wang, P. (Ed.), ISSAC'92, Proc. Int. Symp. on Symbolic Algebraic Comput. ACM Press, Berkeley, pp. 49–56.

Schwarz, F., 1998. ALLTYPES: an algebraic language and type system. In: Calmet, J., Plaza, J. (Eds.), Artificial Intelligence and Symbolic Computation. Lecture Notes in Artificial Intelligence, vol. 1476. Springer, pp. 270–283.

Singer, M., 1991. Liouvillian solutions of linear differential equations with Liouvillian coefficients. J. Symbolic Comput. 11, 251–273.

Tsarev, S., 2001. Factorization of overdetermined systems of linear partial differential equations with finite dimensional solution space. In: Ganzha, V., Mayr, E., Vorozhtsov, V. (Eds.), CASC-2001, Proc. 4th Int. Workshop Computer Algebra Scient. Comput. Springer-Verlag, pp. 529–539.

van der Put, M., Singer, M., 2003. Galois Theory of Linear Differential Equations. Grundleheren der mathematischen Wissenschaften, vol. 328. Springer.

Wu, W.-t., 1989. On the foundation of algebraic differential geometry. Syst. Sci. Math. Sci. 2, 289–312.